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On Surface Waves and Tidal Waves near a Promontory.

BY

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The only problem of the free vibrations of a rotating sheet of gravitating liquid of small uniform depth which may be considered to have been completely solved is the one in which the boundary is circular.[1] When the boundary is not circular, the difficulty of a complete solution is much greater. Lord Rayleigh[2] obtained a partial solution for the case when the boundary is rectangular, applicable when the angular velocity of rotation is small. In a recent paper Proudman[3] has u[illegible] Lord Rayleigh's approximate theory of diffraction to solve a nu[illegible] of other problems, namely, those of the diffraction of a [illegible] wave by an elliptic island, by a semi-elliptic cape, by a [illegible] bay and by a passage between one sea and another.

In the present paper the theory of multiform solution as developed by Sommerfeld has been used to solve the problem of the diffraction of surface waves by a long promontory which for simplicity has been assumed to be either a semi-infinite plane bounded by a straight edge or a wedge forming a definite angle. The nature of the tidal waves on flat rotating sheet of water near a promontory of the above mentioned shapes has also been determined. A method has also been given for determining the free tidal oscillations in a rotating circular sector, a problem[4] which Proudman found to be exceedingly difficult to solve.

[1] Kelvin, *Phil. Mag.*, Aug. 1880; Lamb, *Hydrodynamics*, §§ 208, 209, 210.

[2] Lord Rayleigh, *Phil. Mag.*, V, pp. 297-301 (1903) [*Scientific Papers*, vol. V, p. 93]. See also *Proc. Roy. Soc.*. A. Vol. 82, p. 448.

[3] Proudman, *Proc. Lond. Math. Soc.*, Vol. 14, p. 89 (1915).

[4] *Proc. Lond. Math. Soc.*, Vol. 12, p. 453, (1913).

Case I. *Surface Waves—Diffraction by a semi-infinite plane barrier.*

We shall first consider the case of waves propagated in a liquid of uniform depth under the action of gravity.

Let the plane of the undisturbed surface be the plane of xy, and let the axis of y be measured in the direction of propagation of the wave and the axis of z vertically upwards. Let the barrier occupy the half of the xz plane for which x is positive.

Using cylindrical coordinates, we see that since the motion is supposed to be irrotational, the velocity potential Φ must satisfy the equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}=0. \quad \ldots \quad \ldots \quad (1)$$

At the bottom of the liquid, when $z=-h$,

$$\frac{\partial\Phi}{\partial z}=0. \quad \ldots \quad \ldots \quad (2)$$

On the surface of the barrier, we have

$$\frac{\partial\Phi}{\partial\theta}=0, \text{ when } \theta=0 \text{ and } \theta=2\pi. \quad \ldots \quad \ldots \quad (3)$$

[illegible] free surface the condition to be satisfied is

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}=0, \text{ when } z=0. \quad \ldots \quad \ldots \quad (4)$$

Let us now assume that the incident wave is given by the velocity potential

$$\Phi_0=\text{A}\cosh k(z+h)\cos k(x\sin\theta_1+y\cos\theta_1)\cos pt,$$

where

$$p^2=gk\tanh kh, \quad \ldots \quad \ldots \quad (5)$$

and A is a constant.

It is easy to see that this expression satisfies (1), (2) and (4) and therefore represents a set of plane waves which can be maintained on the surface of an infinite liquid of constant depth h. This set of waves will be incident on the barrier at an angle θ_1. If now Φ represents the velocity potential of the total disturbance when the barrier is present in the liquid, we can assume the following expression for Φ

$$\Phi=\text{A}\cosh k(z+h)f(r,\theta)\cos pt. \quad \ldots \quad \ldots \quad (6)$$

It is now easy to see by substitution in the differential equation (1), that $f(r, \theta)$ must satisfy the equation

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+k^2 f=0 \quad \ldots \quad \ldots \quad (7)$$

and also since on the surface of the barrier $\frac{\partial f}{\partial \theta}=0$ when $\theta=0$ and $\theta=2\pi$, $f(r, \theta)$ must have the same form as the expression obtained by Sommerfeld for the diffraction of plane polarised light by a perfectly reflecting semi-infinite screen, the magnetic force in the incident light being parallel to the edge of the screen. Hence $f(r, \theta)$ must be given by the expression

$$f(r, \theta)=\tfrac{1}{2}\cos k(x\sin\theta_1+y\cos\theta_1)+\tfrac{1}{2}\cos k(x\sin\theta_1-y\cos\theta_1)$$

$$+\tfrac{1}{2}\sqrt{\frac{k}{\pi}}\int_0^{r+x\sin\theta_1+y\cos\theta_1}\cos\left\{\frac{\pi}{4}+k(x\sin\theta_1+y\cos\theta_1-u)\right\}\frac{du}{u}$$

$$+\tfrac{1}{2}\sqrt{\frac{k}{\pi}}\int_0^{r+x\sin\theta_1-y\cos\theta_1}\cos\left\{\frac{\pi}{4}+k(x\sin\theta_1-y\cos\theta_1-u)\right\}\frac{du}{u} \quad \ldots \quad (8)$$

This by means of a result recently obtained by Hargreaves[1] can be written in the form

$$f(r, \theta)=\tfrac{1}{2}\cos k(x\sin\theta_1+y\cos\theta_1)+\tfrac{1}{2}\cos k(x\sin\theta_1-y\cos\theta_1)$$

$$+\left[J_{\frac{1}{2}}(kr)\cos\frac{\theta}{2}\left(\cos\frac{\theta_1}{2}+\sin\frac{\theta_1}{2}\right)+J_{\frac{5}{2}}(kr)\cos\frac{5\theta}{2}\left(\cos\frac{5\theta_1}{2}+\sin\frac{5\theta_1}{2}\right)+\ldots\right]$$

$$+\left[J_{\frac{3}{2}}(kr)\cos\frac{3\theta}{2}\left(\cos\frac{3\theta_1}{2}-\sin\frac{3\theta_1}{2}\right)+J_{\frac{7}{2}}(kr)\cos\frac{7\theta}{2}\left(\cos\frac{7\theta_1}{2}-\sin\frac{7\theta_1}{2}\right)+\ldots\right] \quad \ldots \quad (9)$$

[1] Hargreaves, "A diffraction problem and an asymptotic theorem in Bessel's series," *Phil. Mag.*, Aug., 1918.

For normal incidence $f(r, \theta)$ can be written in the form

$$f(r, \theta)=\cos ky+J_{\frac{1}{2}}(kr)\cos\frac{\theta}{2}+J_{\frac{3}{2}}(kr)\cos\frac{3\theta}{2}+J_{\frac{5}{2}}(kr)\cos\frac{5\theta}{2}$$

$$+J_{\frac{7}{2}}(kr)\cos\frac{7\theta}{2}+\quad\ldots\quad(10)$$

Hence the solution for normal incidence can be written in the form

$$\Phi=A\cosh k(z+h)\cos ky\cos pt$$

$$+A\cosh k(z+h)\left[J_{\frac{1}{2}}(kr)\cos\frac{\theta}{2}+J_{\frac{3}{2}}(kr)\cos\frac{3\theta}{2}+\ldots\right]\cos pt\ldots\quad(11)$$

This expression can be used to plot the stream lines graphically on any plane z=constant. The method is to plot first the equipotential lines Φ=const. starting from equidistant points on the positive side of the x-axis. It will be noticed that the equipotential lines curve round the edge of the barrier and then proceed to infinity asymptotically to the negative direction of the x-axis. The stream lines which consist of the orthogonal set of lines are easily drawn and are found, over a considerable portion of the region, to proceed very nearly from the edge of barrier. This is somewhat analogous to the radiation of light from the edge of the screen in the corresponding optical problem.

Case II. *Surface Waves—Diffraction by a wedge-shaped barrier.*

Let the edge of the wedge be chosen as the axis of z, and let r, θ, z be cylindrical coordinates of a point so that the faces of the wedge are given by $\theta=0$ and $\theta=\alpha$ and the space occupied by it is that between $\theta=\alpha$ and $\theta=2\pi$.

If now the incident wave be represented by the real part of the expression

$$\Phi_0=A\cosh k(z+h)e^{ik[r\cos(\theta-\theta_0)+Vt]}\quad\ldots\quad(12)$$

where $V^2=g/k\tanh kh$,

then it is easy to see that the velocity potential of the total disturbance is given by the real part of the expression

$$\Phi = \frac{A}{2\pi\iota} \cosh k(z+h) e^{ikVt} \int e^{ikr\cos\zeta} \frac{d}{d\zeta} \log(\omega\,\omega_1) d\zeta,$$

where $\omega = \cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta o)$, $\omega_1 = \cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta o)$, ... (13)

the path of integration being a complete contour which starts from $\infty\iota+\lambda$ and goes to $\infty\iota+\lambda$ without crossing the real axis. The expression for Φ can also be written in the form

$$\Phi = \frac{2\pi A}{a} \cosh k(z+h) \left[\sum_1^\infty J_{\frac{n\pi}{a}}(kr) \frac{\pi}{\sin\frac{n\pi^2}{a}} \cos\frac{n\pi\theta}{a} \sin\frac{n\pi\theta_1}{a} \right] \cos kVt \quad (14)$$

As in the previous case this expression may be used to plot the steam lines.[1]

Case III. (1) *Tidal Waves near a long narrow Promontory.*

Suppose the sheet of water to be rotating with uniform angular velocity about an axis perpendicular to its plane and let the depth of water (as rotating in free relative equilibrium) be uniform and equal to h. Let ζ denote the elevation of the free surface at any time. Then for a disturbance in which the time t only enters through the factor $e^{\iota\sigma t}$, we have the equation

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + k^2\zeta = 0, \quad \ldots \quad \ldots \quad (15)$$

there being no disturbing force and

$$k^2 = \frac{\sigma^2 - 4\omega^2}{gh}.$$

The boundary condition is given by

$$\iota\sigma \frac{\partial\zeta}{\partial n} + 2\omega \frac{\partial\zeta}{\partial s} = 0, \quad \ldots \quad \ldots \quad (16)$$

[1] See Wiegrefe, "Über einige mehrwertige Lösungen der Wellengleichung $\nabla^2 u + k^2 u = 0$ und ihre Anwendung in der Beugungstheorie," *Ann. d. Phys.*, Bd. 39, p. 449 (1912).

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward drawn normal to the boundary and $\frac{\partial}{\partial s}$ along the positive direction of the arc. We exclude the cases when $\sigma=0$ and $\sigma^2=4\omega^2$.

For a narrow promontory, the boundary condition reduces to

$$\frac{\iota\sigma}{r}\frac{\partial\zeta}{\partial\theta}+2\omega\frac{\partial\zeta}{\partial r}=0, \quad \ldots \quad \ldots \quad (17)$$

to be satisfied when $\theta=0$ and $\theta=2\pi$.

Let $\zeta=\zeta_1+\zeta_2$;

where ζ_1, ζ_2 are subject to the conditions

$$\frac{\partial\zeta_2}{\partial\theta}=0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial r}=0, \quad \ldots \quad \ldots \quad (18)$$

when $\theta=0$ and $\theta=2\pi$.

When $\sigma^2>4\omega^2$, we can assume the following expressions for ζ_1 and ζ_2;

$$\zeta_1=\sum_{n=o}^{\infty} A_n \, J_{\frac{2n+1}{2}}(kr)e^{\iota\left[\sigma t+\frac{(2n+1)\theta}{2}\right]},$$

$$\zeta_2=\sum_{n=o}^{\infty} \frac{2n+1}{2} \, J_{\frac{2n+1}{2}}(kr)e^{\iota\left[\sigma t+\frac{(2n+1)\theta}{2}\right]}.$$

It is obvious from the conditions (17) and (18) that for ζ_2, we should take the real part of the expression assumed and for ζ_1 the imaginary part and *vice-versa.*

The unknown constant A_n is determined from the boundary condition (17) which gives

$$-\sum\frac{\sigma(2n+1)}{2}A_n\frac{J_{\frac{2n+1}{2}}(kr)}{kr}+\omega\sum(2n+1)\frac{dJ_{\frac{2n+1}{2}}(kr)}{d(kr)}=0 \quad \ldots \quad (19)$$

to be satisfied for all values of r.

But since

$$\frac{2n+1}{z} J_{\frac{2n+1}{2}}(z) = J_{\frac{2n-1}{2}}(z) + J_{\frac{2n+3}{2}}(z),$$

$$\frac{d}{dz} J_{\frac{2n+1}{2}}(z) = \tfrac{1}{2}\left\{ J_{\frac{2n-1}{2}}(z) - J_{\frac{2n+3}{2}}(z) \right\}, \quad \ldots \quad (20)$$

the above equation can be written in the form

$$-\frac{\sigma}{2} \Sigma A_n \left[J_{\frac{2n-1}{2}}(kr) + J_{\frac{2n+3}{2}}(kr) \right]$$

$$+\omega \Sigma (2n+1) \left[J_{\frac{2n-1}{2}}(kr) - J_{\frac{2n+3}{2}}(kr) \right] = 0.$$

Hence equating the co-efficient of $J_{\frac{2n+1}{2}}(kr)$ to zero, we obtain

$$A_{n-1} + A_{n+1} = \frac{8\omega}{\sigma} \quad \ldots \quad \ldots \quad (21)$$

from which we deduce that

$$A_0 = A_1 = A_2 = \ldots = \frac{4\omega}{\sigma} \quad \ldots \quad \ldots \quad (22)$$

Hence the expression for ζ can be written in the form

$$\zeta = \sum_{n=0}^{\infty} J_{\frac{2n+1}{2}}(kr) \left[\frac{4\omega}{\sigma} \sin \frac{(2n+1)\theta}{2} + \frac{2n+1}{2} \cos \frac{(2n+1)\theta}{2} \right] e^{\iota\sigma t} \quad (23)$$

This result will only be applicable over limited portions of the region considered, since the fundamental equations are really only valid for limited regions.

(2) *Free Tidal oscillations in a rotating circular sector bounded by*

$r=a,\ \theta=0,\ \theta=2\pi.$

To obtain the complete solution for the *free* tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$ and $\theta=2\pi$, we notice from the above expression for ζ and from the boundary condition to be satisfied on the circular rim that in the nth mode of vibration k must be taken to be a root of the equation

$$(2n+1)\omega J_{\frac{2n+1}{2}}(ka) + ka\sigma \frac{d}{d(ka)} J_{\frac{2n+1}{2}}(ka) = 0 \quad \ldots \quad (24)$$

Hence the tidal height is given by

$$\zeta=\sum_k \mathrm{B}_k \left[\sum_{n=0}^{\infty} \mathrm{J}_{\frac{2n+1}{2}}(kr)\left\{\frac{4\omega}{\sigma}\sin\frac{(2n+1)\theta}{2}\right.\right.$$

$$\left.\left.+\frac{2n+1}{2}\cos\frac{(2n+1)\theta}{2}\right\}\right] e^{i\sigma_k t},$$

where the summation extends over all the roots of the equation (24) and the constants B_k's are determined by the equations

$$\sum_k \mathrm{B}_k \left[(2s+1)\omega\ \mathrm{J}_{\frac{2s+1}{2}}(ka)+ka\sigma\frac{d}{d(ka)}\mathrm{J}_{\frac{2s+1}{2}}(ka)\right]=0$$

$$[s=0,\ 1,\ 2\ \ldots\ n-1,\ n+1,\ n+2\ \ldots\ ]$$

and the periods of the oscillations in the nth mode are given by k which are the roots of the equation (24).

When $\sigma^2<4\omega^2$, we have to replace $\mathrm{J}_{\frac{2n+1}{2}}(kr)$ by $\mathrm{I}_{\frac{2n+1}{2}}(kr)$.

Case IV. (1) *Tidal Waves near a wedge-shaped Promontory.*

The method given in the previous case can be extended to determine the tidal heights near a wedge-shaped promontory and also the free tidal oscillations in any rotating circular sector.

The boundary condition is

$$\frac{\iota\sigma}{r}\frac{\partial\zeta}{\partial\theta}+2\omega\frac{\partial\zeta}{\partial r}=0$$

to be satisfied when $\theta=0$ and $\theta=\alpha$.

As before we take

$$\zeta=\zeta_1+\zeta_2, \qquad \ldots \qquad \ldots \qquad (25)$$

where ζ_1, ζ_2 are such that

$$\frac{\partial\zeta_2}{\partial\theta}=0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial r}=0$$

when $\theta=0$ and $\theta=\alpha$,

We can assume the following expressions for ζ_1, ζ_2

$$\zeta_1 = \sum_{n=1}^{\infty} A_n J_{\frac{n\pi}{a}}(kr) e^{\iota\left(\sigma t + \frac{n\pi\theta}{a}\right)} \quad \ldots \quad (26)$$

$$\zeta_2 = \sum_{n=1}^{\infty} \frac{n\pi}{a} J_{\frac{n\pi}{a}}(kr) e^{\iota\left(\sigma t + \frac{n\pi\theta}{a}\right)}, \quad \ldots \quad (27)$$

with this restriction that in the final expression we take for ζ_1 the imaginary part and for ζ_2 the real part of the expressions assumed and *vice versa.*

The constant A_n can be determined as in the previous case from the boundary condition

$$\sum \frac{\sigma n\pi}{a} A_n \frac{J_{\frac{n\pi}{a}}(kr)}{kr} + 2\omega \sum \frac{n\pi}{a} \frac{d}{d(kr)} J_{\frac{n\pi}{a}}(kr) = 0, \quad \ldots \quad \ldots \quad (28)$$

which has to be satisfied for all values of r, by the use of the following expansions[1]

$$\frac{J_{\frac{n\pi}{a}}(kr)}{kr} = a_{n-1}\left[J_{\frac{(n-1)\pi}{a}}(kr) + J_{\frac{(n+1)\pi}{a}}(kr)\right]$$

$$+ a_{n-2}\left[J_{\frac{(n-2)\pi}{a}}(kr) + J_{\frac{(n+2)\pi}{a}}(kr)\right] + \text{etc.}, \ldots \quad (29)$$

and
$$\frac{d}{d(kr)} J_{\frac{n\pi}{a}}(kr) = b_{n-1}\left[J_{\frac{(n-1)\pi}{a}}(kr) - J_{\frac{(n+1)\pi}{a}}(kr)\right]$$

$$+ b_{n-2}\left[J_{\frac{(n-2)\pi}{a}}(kr) - J_{\frac{(n+2)\pi}{a}}(kr)\right] + \text{etc.} \ldots \quad (30)$$

$$[2\pi \geqq a \geqq \pi]$$

[1] These expansions do not appear to have been given by any previous writer. A proof of these expansions will be given in a subsequent paper.

(2) *Free Tidal oscillations in a rotating circular sector bounded by*

$r=a,\ \theta=0$, and $\theta=\alpha$.

As in the previous case the method can also be used to determine the free tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$, $\theta=a$.

We have

$$\zeta=\sum_k \mathrm{B}_k \left[\sum_{n=1}^{\infty} \mathrm{J}_{\frac{n\pi}{a}}(kr)\left\{\mathrm{A}_n \sin\frac{n\pi\theta}{a}+\frac{n\pi}{a}\cos\frac{n\pi\theta}{a}\right\}\right] e^{i\sigma_k t}$$

where B_k's are determined by the equations

$$\sum_k \mathrm{B}_k \left[\frac{2\omega s\pi}{a}\,\mathrm{J}_{\frac{s\pi}{a}}(ka)+\sigma ka\,\frac{d}{d(ka)}\,\mathrm{J}_{\frac{s\pi}{a}}(ka)\right]=0$$

$[s=1, 2, \ldots n-1, n+1, n+2 \ldots]$

and the summation extends over all the values of k which are the roots of the equation—

$$\frac{2\omega n\pi}{a}\,\mathrm{J}_{\frac{n\pi}{a}}(ka)\Big/ka+\sigma\,\frac{d}{d(ka)}\,\mathrm{J}_{\frac{n\pi}{a}}(ka)=0.$$

Hence the periods of the free tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$, $\theta=a$ in the nth mode are given by k which are the roots of the equation

$$\frac{2\omega n\pi}{d}\cdot\frac{\mathrm{J}_{\frac{n\pi}{a}}(ka)}{ka}+\sigma\,\frac{d}{d(ka)}\,\mathrm{J}_{\frac{n\pi}{a}}(ka)=0.$$

On the potentials of uniform and heterogeneous elliptic cylinders at an external point.

BY

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1.

The potential of an infinite elliptic cylinder at an external point is generally expressed in the form of an integral and it is well-known that a transformation in conjugate functions would allow the same integral to be represented by a much simpler expression.[1] It is here proposed to express the potential in trigonometrical series. The method followed is that of integration which will be shown to be applicable also in the case of heterogeneity. It will be found that the potential is always expressible as

$$A_0 \log r - \sum_{n=1}^{\infty} A_n \frac{\cos n\theta}{r^n}$$

where A_n in its most general form can be expressed by hypergeometric functions in e^2 (eccentricity), reducing in two special cases to finite binomial forms. This happens when the cylinder is homogeneous or when the density (supposed constant along lines parallel to the axis) at any point on the elliptic section varies inversely as the focal distance of the point. We shall simplify our problem by considering only the logarithmic potential of the elliptic section to which the (Newtonian) potential of the elliptic cylinder is equivalent but for an infinite constant and the constant multiplier 2.

2.

Before proceeding with the solution of the problem proposed above it would be useful to consider the expansion of $(1+e\cos\phi)^{-n}$, $e<1$, in cosines of multiples of ϕ. Expanding $(1+e\cos\phi)^{-n}$ by the binomial

[1] Lamb, Mess. of Math. 1878.

theorem and replacing the powers of $\cos\phi$ by cosines of multiples of ϕ it may be shown that[1]

$$(1+e\cos\phi)^{-n} = \sum_{m=0}^{\infty} A^n_m \cos m\phi$$

where

$$A^n_m = (-1)^m \frac{(n+m-1)!}{(n-1)!\,m!} \frac{e^m}{2^{m-1}} F\left(\frac{n+m}{2}, \frac{n+m+1}{2}, m+1, e^2\right),$$

$$(e<1)$$

where F is the hypergeometric function of the four elements within the parenthesis. Since A^n_m is a Fourier's co-efficient, we have

$$\int_{-\pi}^{\pi} \frac{\cos m\phi\, d\phi}{(1+e\cos\phi)^n} = \pi A^n_m$$

$$= (-1)^m \frac{(n+m-1)!}{(n-1)!\,m!} \frac{\pi e^m}{2^{m-1}} F\left(\frac{n+m}{2}, \frac{n+m+1}{2}, m+1, e^2\right)$$

and

$$\int_{-\pi}^{\pi} \frac{\cos n\phi\, d\phi}{(1+e\cos\phi)^{n+2}} = (-1)^n \frac{(2n+1)!}{(n+1)!\,n!} \frac{\pi e^n}{2^{n-1}} F\left(n+1, \frac{2n+3}{2}, n+1, e^2\right)$$

$$= (-1)^n \frac{(2n+1)!}{(n+1)!\,n!} \frac{\pi e^n}{2^{n-1}} \frac{1}{(1-e^2)^{n+\frac{3}{2}}};$$

also

$$\int_{-\pi}^{\pi} \frac{\cos n\phi\, d\phi}{(1+e\cos\phi)^{n+1}} = (-1)^n \frac{(2n)!}{n!\,n!} \frac{\pi e^n}{2^{n-1}} F\left(n+\tfrac{1}{2}, n+1, n+1; e^2\right)$$

$$= (-1)^n \frac{(2n)!}{n!\,n!} \frac{\pi e^n}{2^{n-1}} \frac{1}{(1-e^2)^{n+\frac{1}{2}}},$$

n being an integer and $e<1$.

[1] This expansion in another form is given by Gauss.

3.

Taking the focus S as origin let the equation of the ellipse be given by

$$\rho = \frac{l}{1+e\cos\phi}.$$

Let P be any point $(\rho.\ \phi)$ within the area of the ellipse and A another point $(r,\ \theta)$ at a sufficient distance from it. Taking the area to be of unit density the potential

$$V = \int_{-\pi}^{\pi}\int_{0}^{\frac{l}{1+e\cos\phi}} \log AP\ \rho d\rho d\phi.$$

Now

$$AP^2 = \rho^2 - 2\rho r\cos(\phi-\theta) + r^2$$

and

$$\log AP = \log r + \tfrac{1}{2}\log\left[1-2\left(\frac{\rho}{r}\right)\cos(\phi-\theta) + \left(\frac{\rho}{r}\right)^2\right]$$

$$= \log r - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\phi-\theta),$$

r being supposed to be greater than the maximum radius vector, *i.e.*, the length of the major axis from S to the remoter vertex of the ellipse.

Hence

$$V = \frac{l^2}{2}\int_{-\pi}^{\pi}\frac{d\phi}{(1+e\cos\phi)^2}\log r - \sum_{n=1}^{\infty}\frac{l^{n+2}}{n(n+2)r^n}\int_{-\pi}^{\pi}\frac{\cos n(\phi-\theta)d\phi}{(1+e\cos\phi)^{n+2}}.$$

But

$$\int_{-\pi}^{\pi}\frac{\cos n\phi\ d\phi}{(1+e\cos\phi)^{n+2}} = (-1)^n\frac{(2n+1)!}{(n+1)!\,n!}\frac{\pi e^n}{2^{n+1}}\frac{1}{(1-e^2)^{n+\frac{3}{2}}}$$

and

$$\int_{-\pi}^{\pi}\frac{\sin n\phi\ d\phi}{(1+e\cos\phi)^{n+2}} = 0.$$

Hence

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}V = \log r - \sum_{n=1}^{\infty}(-1)^n \frac{(2n+1)!}{n.\,n!\,(n+2)!}\frac{1}{2^{n-1}}\left(\frac{le}{1-e^2}\right)^n\frac{\cos n\theta}{r^n}.$$

But

$$\frac{le}{l-e^2} = ae = \mathrm{CS} = c,$$

where C is the centre.

So that we have finally

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}\,V = \log r - \sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)!}{n.\,n!\,(n+2)!}\frac{1}{2^{n-1}}\left(\frac{c}{r}\right)^n\cos n\theta$$

and 2V is the potential of the elliptic cylinder neglecting an infinite constant.

4.

Let us suppose the cylinder to be heterogeneous and any line parallel to the axis to be a line of equal density. Let the density at the point (x, y) on the elliptic section be $f(x, y)$ where f is a rational algebraic integral function in x and y. Such a function is also expressible in a series in ρ and ϕ of which the typical terms are $\rho^p\cos q\phi$ and $\rho^p \sin q\phi$. It will be sufficient for us to work out the case of these two densities.

(i) Suppose the density $\sigma = \rho^p \cos q\phi$.

Then as before

$$V = \iint \sigma \log \mathrm{AP}\, \rho d\rho d\phi,$$

the integration is to be carried over the entire area of the ellipse. Proceeding exactly as in the previous case we have

$$V = \frac{l^{p+2}}{(p+2)}\int_{-\pi}^{\pi}\frac{\cos q\phi\, d\phi}{(l+e\cos\phi)^{p+2}}\log r - \sum_{n=1}^{\infty}\frac{l^{n+p+2}}{n(n+p+2)r^n}\int_{-\pi}^{\pi}\frac{\cos q\phi \cos n(\phi-\theta)d\phi}{(1+e\cos\phi)^{n+p+2}}$$

$$= \frac{l^{p+2}}{(p+2)}\int_{-\pi}^{\pi}\frac{\cos q\phi\, d\phi}{(1+e\cos\phi)^{p+2}}\log r - \sum_{n=1}^{\infty}\frac{l^{n+p+2}\cos n\theta}{2^n(n+p+2)r^n}$$

$$\int_{-\pi}^{\pi}\frac{[\cos(n+q)\phi + \cos(n-q)\phi]\,d\phi}{(1+e\cos\phi)^{n+p+2}}$$

and substituting the values of the integrals from § 2

$$\frac{V}{\pi l^{p+2}}=\frac{A_q^{p+2}}{(p+2)}\log r-\sum_{n=1}^{\infty}\frac{A_{n+q}^{n+p+2}+A_{\overline{n-q}}^{n+p+2}}{2n(n+p+2)}\left(\frac{l}{r}\right)^n\cos n\theta,$$

where $\overline{n-q}$ is the positive value of the difference between these two integers.

When $q=p$ one half of the series is expressible in a simpler form; since

$$A_{n+p}^{n+p+2}=(-1)^{n+p}\frac{(2n+2p+1)!}{(n+p+2)!\,(n+p)!}\,\frac{e^{n+p}}{2^{n+p-1}}\,\frac{1}{(1-e^2)^{n+p+\frac{3}{2}}}$$

this part of the potential function can be written as

$$\frac{(-1)^p e^p}{2^p\cdot(1-e^2)^{p+\frac{3}{2}}}\left[\frac{2\cdot(2p+1)!}{p!(p+2)!}\log r-\sum_{n=1}^{\infty}(-1)^n\frac{(2n+2p+1)!}{n\cdot(n+p+2)!\,(n+p)!}\left(\frac{c}{2r}\right)^n\cos n\theta\right]$$

the other part being expressed in hypergeometric functions. This is possible only within the limits in which such separation of terms is legitimate. A similar simplification is possible when $q=p+1$.

When $q=0$ and $\sigma=\rho^p$ the potential is given by

$$\frac{V}{\pi l^{p+2}}=\frac{A_0^{p+2}}{(p+2)}\log r-\sum_{n=1}^{\infty}\frac{A_n^{n+p+2}}{n(n+p+2)}\left(\frac{l}{r}\right)^n\cos n\theta.$$

It may be noted that when p is a negative integer this formula is applicable with a slight modification. The terms beginning from the first up to the nth where $n=-p+1$ would have their co-efficients in finite forms which it is easy to calculate.

A very interesting case of the above arises when $p=-1$ or the heterogeneity is of such a nature that the density at any point on the elliptic section varies inversely as the focal distance. Since

$$A_n^{n+1}=(-1)^n\frac{(2n)!}{n!\,n!}\,\frac{e^n}{2^{n-1}}\cdot F(n+\tfrac{1}{2},\,n+1,\,n+1;\,e^2)$$

$$=(-1)^n\frac{(2n)!}{n!\,n!}\,\frac{e^n}{2^{n-1}}\,\frac{1}{(1-e^2)^{n+\frac{1}{2}}};$$

we have in such a case

$$\frac{V}{\pi l} = A^1_0 \log r - \sum_{n=1}^{\infty} \frac{A^{n+1}_n}{n(n+1)} \left(\frac{l}{r}\right)^n \cos n\theta$$

and making the above substitutions the potential function is found to be given by

$$\frac{V(1-e^2)^{\frac{1}{2}}}{\pi l} = 2\log r - \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^{n-1} n.n!(n+1)!} \left(\frac{c}{r}\right)^n \cos n\theta.$$

(ii) Suppose that the density $\sigma = \rho^p \sin q\phi$.

Then as before

$$V = \frac{l^{p+2}}{(p+2)} \int_{-\pi}^{\pi} \frac{\sin q\phi \, d\phi}{(1+e\cos\theta)^{p+2}} \log r - \sum_{n=1}^{\infty} \frac{l^{n+p+2}}{n(n+p+2)} \int_{-\pi}^{\pi} \frac{\sin q\phi \cos n(\phi-\theta) d\phi}{(1+e\cos\phi)^{n+p+2}}.$$

Now $$\int_{-\pi}^{\pi} \frac{\sin q\phi \, d\phi}{(1+e\cos\phi)^{p+2}} = 0$$

and replacing the product in the numerator of the other integral by the sum of two sines we have

$$\frac{V}{\pi l^{p+2}} = -\sum_{n=1}^{\infty} \frac{A^{n+p+2}_{n-q} - A^{n+p+2}_{n+q}}{2n(n+p+2)} \left(\frac{l}{r}\right)^n \sin n\theta.$$

The logarithmic term is absent. The line $\theta=0$, $\theta=\pi$ is a line of zero potential, as is obvious. Also at a great distance from the origin where $\frac{1}{r^2}, \frac{1}{r^3}$ etc. can be neglected the potential is approximately given by

$$V = -\frac{\pi l^{p+2}}{2(p+3)} \left[A^{p+3}_{q-1} - A^{p+3}_{q+1}\right] \frac{\sin\theta}{r};$$

hence the corresponding equipotential lines are arcs of circles touching the major axis at the focus.

5.

We can determine all the cases in which the hypergeometric functions appearing as co-efficients in the trigonometrical series are expressible in finite forms. Two cases we have already studied where

they reduce into binomials; let us enquire if in any other cases such reduction is possible. The function F $(\alpha; \beta, \gamma, e^2)$ will be a binomial expansion if either $\gamma=\alpha$ or $\gamma=\beta$.* Taking the most general case (*i*) § 4 we seek to satisfy either of these conditions in both the functions A^{n+p+2}_{n+q} and A^{n+p+2}_{n-q} by giving suitable values to p and q.

$$A^{n+p+2}_{n+q} = C\times F\left(n+1+\frac{p+q}{2},\ n+\frac{3}{2}+\frac{p+q}{2},\ n+q+1;\ e^2\right),$$

hence for the required condition we should have

$$n+q+1=\text{either}\quad n+1+\frac{p+q}{2}$$

$$\text{or}\qquad n+\frac{3}{2}+\frac{p+q}{2},$$

$$\text{i.e.,}\qquad \left.\begin{aligned} &p=q\\ \text{or}\quad &p+1=q \end{aligned}\right\}$$

and $A^{n+p+2}_{n-q} = C'\times F\left(n+1+\frac{p-q}{2},\ n+\frac{3}{2}+\frac{p-q}{2},\ n-q+1;\ e^2\right)$

which in a similar manner gives

$$\left.\begin{aligned} p&=3\,q\\ p+1&=3\,q \end{aligned}\right\}.$$

So we are to find p and q such that any of the four following sets of equations should be consistent

$$\left.\begin{aligned} p&=q\\ p&=3q \end{aligned}\right\}\quad \left.\begin{aligned} p&=q\\ p+1&=3q \end{aligned}\right\}\quad \left.\begin{aligned} p+1&=q\\ p&=3q \end{aligned}\right\}\quad \left.\begin{aligned} p+1&=q\\ p+1&=3q \end{aligned}\right\}.$$

q being zero or an integer.

From these four equations we get only two possible solutions namely

$$\left.\begin{aligned} p&=o\\ q&=o \end{aligned}\right\}\qquad \left.\begin{aligned} p&=-1\\ q&=o \end{aligned}\right\}$$

* The other complicated forms, *e.g.*, $F\left\{\frac{1}{2}+\frac{1}{2n}, \frac{1}{2n}, 1+\frac{1}{n}; x^2\right\}$ are at once seen to be inapplicable here.

answering to the case of homogeneity and to that of the density varying as the inverse focal distance. These are the only two cases in which the potential function for an infinite elliptic cylinder for the outside space is expressible in a trigonometrical series with binomial co-efficients.

6.

In § 3 we have obtained a trigonometrical series for the potential function V for the outside space by integrating log r throughout the entire area of the section. But in course of our analysis in order to make the expansion of the logarithim of the distance PA possible we had to introduce a certain limitation, namely that r should always be greater than ρ, this immediately marks out a circular area with centre S and radius equal to the maximum radius vector within which the point A must not lie. It will now be shown that the series V has a much wider area of convergence which extends even into the limiting circle and consequently from considerations of continuity it represents the potential function everywhere inside that extended area.

It is well-known that the series $\Sigma(-1)^n\ a_n \cos n\theta$ is convergent if $a_n \longrightarrow o$ steadily. Considering the present series as a series of the same type we have

$$a_n = \frac{2\ (2n+1)\ !}{n.\ n\ !\ \ (n+2)\ !}\ \left(\frac{c}{2r}\right)^n$$

$$= \frac{2}{n(n+2)} \cdot \frac{1.\ 3.\ 5 \ldots (2n+1).\ 2^n.\ n\ !}{n\ !\ \ (n+1)\ !}\ \left(\frac{c}{2r}\right)^n$$

$$= \frac{2^2}{n(n+2)} \cdot \frac{1.\ 3.\ 5 \ldots (2n+1)}{2.\ 4.\ 6 \ldots (2n+2)} \cdot \left(\frac{2c}{r}\right)^n$$

and this would be a decreasing monotonous sequence tending towards the limit zero if we take

$$r \geqq 2c$$

$$\geqq SS',$$

where S′ is the second focus. We can also show by applying the usual ratio-test that the series is absolutely convergent under the same conditions. This shows that in addition to the outside region the series is also convergent inside the area lying between the previous limiting circle (drawn for the purpose of integration) and a concentric circle whose radius is SS′. Consequently the present form of the

potential function is valid at all points inside these two circles (and outside the elliptic area as we are dealing only with the external potential).

It would seem that we are incapable of accepting the potential function in the present form of the infinite series inside the circle of radius SS′. But in fact the region in which this trigonometrical series fails is much more limited. If we take S′ as our origin and proceed to find the potential by the present method we get the same series which in a similar manner may be shown to be applicable everywhere outside a circle of radius SS′. In general these two circles bounding the regions of convergence overlap outside the elliptic area and it is only inside the two small areas common to the two circles and symmetrical about the minor axis that the present trigonometrical series fails. Excepting this common portion the present form of V would hold good everywhere only we should take care to choose the origin properly—measuring r from S or S′ according as the point lies inside the circle of centre S′ or S.

It is curious to note that the convergence of the series depends on the eccentricity of the ellipse. The two limiting circles would have their common portion entirely within the elliptic area if

$$SS' \leqq SB, \text{ where B is an extremity of the minor axis ;}$$

$$i.e., \quad 2ae \leqq a$$

$$i.e., \quad e \leqq \tfrac{1}{2}.$$

This shows that when the eccentricity of the ellipse is not greater than $\frac{1}{2}$ the function V gives the potential everywhere outside the elliptic area, with judicious choice of origin. This includes the important case when the ellipticity is small and the ellipse is obtained from a circle by a slight deformation,

For the area within the two circular strips in which the trigonometrical series fails it is not possible to get by the present method a simple value for the potential function V. Starting from the beginning, we have to divide the elliptic area into two areas by a circle passing through the point where the potential is sought such that every part of the one area is nearer to the origin than the point while every part of the second area is further from it. We can use two logarithmic expansions in the two areas and find out the potential of the two areas separately. The method of procedure is the same as

in § 9. We get the potential both in direct and inverse powers of r. But as the expression is not a simple one we do not propose to give it here.

7.

A similar investigation is possible in the case of the variable density. When the heterogeneity is of the nature we have assumed in § 4 we can show that at least outside the same two strips of areas between the two limiting circles the series V in § 4 is convergent. It should be noticed that a transfer of origin to the other focus in the case of heterogeneity would entail a change in the law of density. But if we take the density to be a rational, algebraic, integral function of the co-ordinates of a point a transfer of origin would involve a change of density of such a nature that the new distribution would still be represented by terms of the form $\rho^p \cos q\phi$ and $\rho^p \sin q\phi$. So these two cases are sufficient for our purpose.

As before applying Dirichlet's test to V in § 4 we get the condition of convergence by making the co-efficient of $\cos n\theta$ steadily tend to zero. This leads to such a condition as the following

$$\operatorname*{Lt}_{n \longrightarrow \infty} \frac{(2n+p+q+1)\,!}{n.\ (n+p+2)\,!\ (n+q)\,!} \frac{e^{n+q}}{2^{n+q}}.$$

$$F(n+1+\frac{p+q}{2}, n+\tfrac{3}{2}+\frac{p+q}{2}, n+q+1, e^2) \left(\frac{l}{r}\right)^n \longrightarrow 0.$$

If $p \leqq q$ every term of F is less than the corresponding term in the expansion of $\dfrac{1}{(1-e^2)^{n+\frac{3}{2}+\frac{p+q}{2}}}$; so that

$$F < \frac{1}{(1-e^2)^{n+\frac{3}{2}+\frac{p+q}{2}}}.$$

Let $p > q$; the hypergeometric series is of the form

$$F\ (\alpha, \alpha+\tfrac{1}{2}, \gamma\ ;\ e^2) = 1 + \frac{\alpha(\alpha+\frac{1}{2})}{1.\gamma} e^2 + \frac{\alpha(\alpha+1)(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})}{1.\ 2.\ \gamma(\gamma+1)} e^4 + \ldots$$

Since $\alpha > \gamma$ (q being positive)

$$\frac{\alpha}{\gamma} > \frac{\alpha+1}{\gamma+1} > \frac{\alpha+2}{\gamma+2} > \text{ etc };$$

so $$F < 1+\frac{a+\frac{1}{2}}{1}\cdot\left(\frac{a}{\gamma}e^2\right)+\frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2\,!}\cdot\left(\frac{a}{\gamma}e^2\right)^2+\dots\dots\dots$$

$$< \frac{1}{\left(1-\frac{a}{\gamma}e^2\right)^{a+\frac{1}{2}}}$$

when the series is convergent.

Here $\mathrm{Lt}\,\frac{a}{\gamma}=1$; we can show as in § 6 that V should converge at least (whatever p and q may be) if

$$\underset{n\longrightarrow\infty}{\mathrm{Lt}}\quad C\times\left[\frac{2el}{r(1-e^2)}\right]^n\longrightarrow 0$$

where C is ultimately of the order $\frac{1}{n^2}$; if

$$r > \frac{2le}{1-e^2}$$

i.e., $$> 2c.$$

Hence, at least outside the same restricted region as in § 6, V represents the potential function for the whole external space.

8.

In § 3 let us suppose that e is equal to zero. An ellipse of zero eccentricity is a circle and the semi-latus rectum is the radius. Making this substitution we have the logarithmic potential of a circular area

$V=\pi a^2\log r=$(area of the circle)$\times$(log of the distance from the centre) and the potential of an infinite circular cylinder is twice this quantity neglecting an infinite constant. Similarly from § 4 when the density varies as the inverse focal distance we have

V=(circumference of the circle)$\times$(log of the distance from the centre) and the potentials of heterogeneous circular cylinders can in the same way be deduced from the other formulae in § 4. Of course all these results admit of easy verification by direct integration.

We shall deduce another simple result from the series for the potential function in § 3. Let us calculate the attraction of the elliptic cylinder at a point on the major—axis produced of the section. On the major axis

$$\frac{\partial V}{\partial \theta}=0$$

and the attraction is $2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}$.

Differentiating the series of §3 we have

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=\frac{1}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)\,!}{n\,!\,(n+2)\,!}\frac{1}{2^{n-1}}\frac{c^n}{r^{n+1}}$$

$$2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=4\pi ab\left[\frac{1}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{2(2n+1)\,!}{n\,!\,(n+2)\,!}\left(\frac{c}{2}\right)^n\frac{1}{r^{n+1}}\right]$$

$$=4\pi\frac{ab}{c^2}r\left[\frac{1}{2\,!}\frac{c^2}{r^2}-\frac{1.3}{3\,!}\frac{c^3}{r^3}+\frac{1.3.5}{4\,!}\frac{c^4}{r^4}-\dots\quad\dots\right]$$

$$=4\pi\frac{ab}{c^2}r\left[\left(1+\frac{c}{r}\right)-\sqrt{1+2\frac{c}{r}}\right]$$

$$=4\pi\frac{ab}{c^2}\left[(c+r)-\sqrt{(c+r)^2-c^2}\right]$$

$$=4\pi\frac{ab}{c^2}\left[\xi-\sqrt{\xi^2-c^2}\right]$$

where ξ is the distance of the point from the centre of the ellipse. This is the total attraction of an infinite elliptic cylinder at an external point on the major axis of its section, a very well-known result.

It is also interesting to note that when the cylinder is heterogeneous, the density at any point of the section varying as the inverse focal distance, the attraction at any point on the major axis is similarly expressible in a very simple form. Using the corresponding formula of § 4 we have as before $\frac{\partial V}{\partial \theta}=0$ on the major axis ;

and

$$\frac{(1-e^2)^{\frac{1}{2}}}{\pi l}\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=\frac{2}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!\,(n+1)!}\cdot\frac{1}{2^{n-1}}\,\frac{c^n}{r^{n+1}}$$

$$=\frac{2}{r}+\frac{2}{c}\sum_{n=1}^{\infty}(-1)^n\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n+2)}\left(\frac{2c}{r}\right)^{n+1}$$

$$=\frac{2}{r}+\frac{2}{c}\left[\left(1+\frac{2c}{r}\right)^{\frac{1}{2}}-\left(1+\frac{c}{r}\right)\right]$$

Hence the total attraction $=2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}$

$$=4\pi\frac{b}{c}\left[\left(1+\frac{2c}{r}\right)^{\frac{1}{2}}-1\right]$$

$$=4\pi\frac{b}{c}\left[\sqrt{\frac{\xi+c}{\xi-c}}-1\right]$$

where ξ is the distance of the point from the centre of the ellipse.

9.

We have so far considered the case of the complete elliptic area. The method of analysis followed here is, however, applicable to the case of an area bounded by two elliptic arcs. As any two arbitrary arcs would make the results cumbrous we choose here for illustration a very simple case when the result appears in a rather symmetrical form. Let us suppose that the two elliptic arcs have the same focus and their major axes lie along the same line. Let S be the common focus and let the two arcs whose equations are

$$\rho=\frac{l}{1+e\cos\phi}$$

and
$$\rho=\frac{l'}{1+e'\cos\phi'}$$

intersect at C and D and let CS make an angle β with the line from which ϕ is measured. P is any point (ρ,ϕ) inside the area and A another point (r,θ) outside at a sufficient distance from the focus. The area

is divided into two elliptic sectors by the radii vectores SC and SD and the potential of the whole area is the sum of the potentials V_1 and V_2 due to the two sectors. We shall suppose the area to be of unit density. Then

$$V_1=\int_{-\beta}^{\beta}\int_{0}^{\frac{l}{1+e\cos\phi}} \log AP\ \rho d\rho d\phi$$

$$=\int_{-\beta}^{\beta}\int_{0}^{\frac{l}{1+e\cos\phi}} \left[\log r-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\phi-\theta)\right] \rho d\rho d\phi$$

r being supposed to be greater than SC and the point A lying outside the limiting circle described with centre S and radius SC. Hence

$$V_1=\frac{l^2}{2}\int_{-\beta}^{\beta}\frac{d\phi}{(1+e\cos\phi)^2}\log r-\sum_{n=1}^{\infty}\frac{l^{n+2}}{n(n+2)r^n}\int_{-\beta}^{\beta}\frac{\cos n(\phi-\theta)\,d\phi}{(1+e\cos\phi)^{n+2}}$$

or, $$\frac{V_1}{l^2}=I_0(\beta,e)\log r-2\sum_{n=1}^{\infty}\frac{I_n(\beta,e)}{n(n+2)}\left(\frac{l}{r}\right)^n\cos n\theta,$$

where $$I_n(\beta,e)=\int_0^{\beta}\frac{\cos n\phi\, d\phi}{(1+e\cos\phi)^{n+2}}$$

and $$I_0(\beta,e)=\int_0^{\beta}\frac{d\phi}{(1+e\cos\phi)^2}$$

$$=\frac{2}{(1-e^2)^{\frac{3}{2}}}\tan^{-1}\left[\tan\frac{\beta}{2}\sqrt{\frac{1-e}{1+e}}\right]-\frac{e}{1-e^2}\cdot\frac{\sin\beta}{1+e\cos\beta}.$$

We shall put $I_n(\beta,e)$ in the form of a series. Putting

$$(1+e\cos\phi)^{-\overline{n+2}}=\sum_{m=0}^{\infty}A_m^{n+2}\cos m\phi$$

where $$A_m^{n+2}=(-1)^m\frac{(n+m+1)!}{(n+1)!\,m!}\,\frac{e^m}{2^{m-1}}F\left(\frac{n+m+2}{2},\frac{n+m+3}{2},m+1;e^2\right),$$

$(e<1)$ § 2.

Since $$\int_0^\beta \cos m\phi \cos n\phi \, d\phi = \tfrac{1}{2}\left[\frac{\sin(m+n)\beta}{m+n} + \frac{\sin(m-n)\beta}{m-n}\right]$$

$$\equiv \tfrac{1}{2} S_{m,n}$$

$$I_n(\beta,e) = \int_0^\beta \frac{\cos m\phi \, d\phi}{(1+e\cos\phi)^{n+2}}$$

$$= \tfrac{1}{2}\sum_{m=0}^{\infty} A_m^{n+2} S_{m,n};$$

we have

$$\frac{V_1}{l^2} = I_0(\beta,e)\log r - \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m^{n+2} S_{m,n}}{n(n+2)}\left(\frac{l}{r}\right)^n \cos n\theta.$$

Similarly

$$\frac{V_2}{l'^2} = I'_0(\pi-\beta,e')\log r - \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+1}\frac{A_m'^{n+2} S_{m,n}}{n(n+2)}\left(\frac{l'}{r}\right)^n \cos n\theta.$$

The potential of the complete area

$$V = V_1 + V_2.$$

It should be observed that the quantities l, l', β, e, e' are not all independent; in fact β is determined by the equation

$$\frac{l}{1+e\cos\beta} = \frac{l'}{1-e'\cos\beta}.$$

When the point A lies within the limiting circle an analysis on the same lines is possible if we divide the elliptic area into two parts (by a circle with S as centre and SA as radius) in which two separate logarithmic expansions would apply. In the most general case of two arbitrary elliptic arcs, the area may be looked upon as the sum of two elliptic segments each of which is the difference of an elliptic sector and a triangle. In the preceding analysis we have virtually given the potential of an elliptic sector and the potential of a triangle is known. But as the result in all these cases are not simple or symmetrical it is unnecessary to deal with them here.

10.

In this connection we may also study the potential of the complete cylinder when the density is an exponential function of the vectorial angle ϕ. As will be shown below this may be considered as a generalisation of the preceding cases. The method of analysis followed would be exactly similar.

Suppose $\sigma = e^{k\phi}$.

Then

$$V=\int_{-\pi}^{\pi}\int_{0}^{\frac{l}{1+\epsilon\cos\phi}} e^{k\phi}\left[\log r-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{r}\right)^{n}\cos n(\phi-\theta)\right]\rho d\rho d\phi$$

$$=\frac{l^2}{2}\int_{-\pi}^{\pi}\frac{e^{k\phi}\,d\phi}{(1+\epsilon\cos\phi)^2}\log r-\sum_{n=1}^{\infty}\frac{l^{n+2}}{n(n+2)r^n}\int_{-\pi}^{\pi}\frac{e^{k\phi}\cos n(\phi-\theta)d\phi}{(1+\epsilon\cos\phi)^{n+2}}.$$

Now

$$\int_{-\pi}^{\pi}\frac{e^{k\phi}\cos n(\phi-\theta)}{(1+\epsilon\cos\phi)^{n+2}}\,d\phi=\tfrac{1}{2}\int_{-\pi}^{\pi}\sum A_m^{n+2}\,e^{k\phi}\left[\cos(\overline{n+m}\,\phi-n\theta)\right.$$

$$\left.+\cos(\overline{n-m}\,\phi-n\theta)\right]d\phi.$$

and

$$\int_{-\pi}^{\pi}e^{k\phi}\cos(\overline{n+m}\,\phi-n\theta)\,d\phi$$

$$=(-1)^{n+m}\,2.\,\frac{k\cos n\theta-(n+m)\sin n\theta}{k^2+(n+m)^2}\sinh k\pi,$$

and

$$\int_{-\pi}^{\pi} e^{k\phi} \cos\,(\overline{n-m}\,\phi - n\theta)\; d\phi$$

$$= (-1)^{n-m} \quad 2. \; \frac{k\cos n\theta - (n-m)\sin n\theta}{k^2+(n-m)^2} \sinh k\pi$$

Hence

$$\frac{\mathrm{V}}{l^2 \sinh k\pi} = \sum_{m=o}^{\infty} (-1)^m \frac{2k\;\mathrm{A}_m^2}{k^2+m^2} \log r - \sum_{n=1}^{\infty} \sum_{m=o}^{\infty} (-1)^{n+m}$$

$$\left[\frac{k\cos n\theta-(n+m)\sin n\theta}{k^2+(n+m)^2} + \frac{k\cos n\theta-(n-m)\sin n\theta}{k^2+(n-m)^2}\right] \times \frac{\mathrm{A}_m^{n+2}}{n(n+2)}\left(\frac{l}{r}\right)^n.$$

If we put $k=o$ the cylinder becomes homogeneous and the present series in this limiting case degenerates into the series of § 3. Moreover the case of the density $\rho\mu e^{k\phi}$ can be easily worked out in a similar manner and putting $ik(i=\sqrt{-1})$ for k we can deduce the formulae of § 4. Thus this form appears to embody in itself all the preceding different cases.

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Notes on Inversion.

BY

TARAKNATH BHATTACHARYYA.

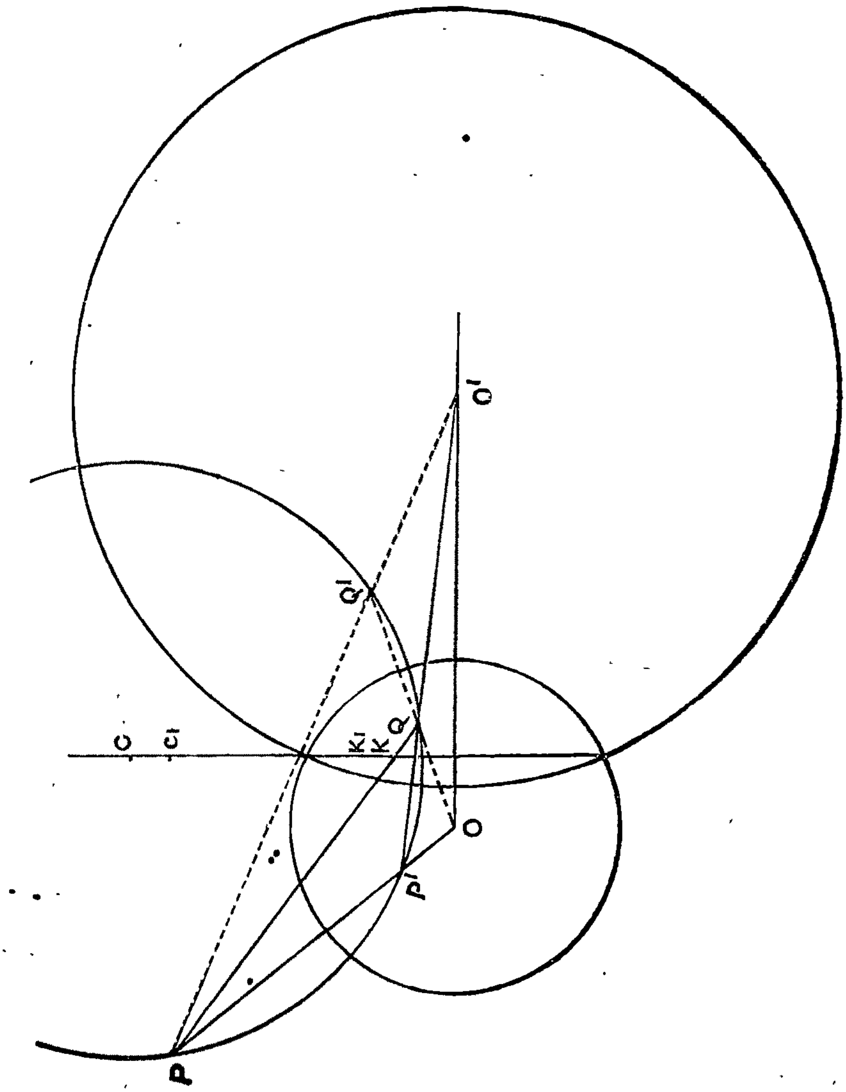
I

1. *Introductory remarks.*—The inverse of a circle with respect to any origin O is a circle; but if the circle of inversion intersects the circle orthogonally, it becomes its own inverse, the points being redistributed. Thus to find the inverse of any point P, we describe a circle through P cutting the circle of inversion orthogonally, and find the point where OP intersects this circle. Thus, "All circles that pass through a fixed point and cut a given circle orthogonally must pass through a second fixed point."

If a quadrangle be inscribed in a conic, its diagonal triangle is self-polar with respect to the conic. If however the diagonal triangle be given, an infinite number of such quadrangles can be drawn. Hence "If an infinity of triangles be inscribed in a conic so that their sides may pass through three fixed points, these fixed points will form a triangle self-polar with respect to the conic." When the conic is a circle, its centre is at the orthocentre of the triangle formed by the fixed points.

2. *Theorem.*—The inverse of a point P with respect to two given orthogonal circles in succession is a fixed point Q *which is independent of the order in which the circles are taken.*

Thus, let (O) and (O′) be two circles cutting one another orthogonally and P any point in the plane. Describe a circle (C) through P cutting the given circles orthogonally. Join OP cutting (C) in P′, and let O′P′ cut it again in Q, which is thus the inverse of P with respect to the given circles. Now PP′Q is a triangle inscribed in a circle and two of its sides pass through O and O′. Therefore, by § 1, PQ will pass through a third fixed point K, the orthocentre of the triangle COO′. Thus Q must be necessarily determinate. Similar remarks apply for the triangle PQQ′. Hence the inverse of a point with respect to two orthogonal circles in succession, taken twice over, is the point itself.


O'
Q'
C
C'
K'
K
Q
O
P'
P

3. We may go a step further. We can invert with respect to *four* mutually orthogonal circles and the result will be *the same.* But if four circles cut one another orthogonally, the centre of any one of them is (the radical centre and hence) the orthocentre of the triangle formed by the centres of the other three circles. One of these circles, therefore, must be imaginary. Thus a glance at the adjoining figure is sufficient to reveal the theorem : "The inverse of a point P with respect to the four mutually orthogonal circles O, O′, C, K in succession is the original point itself : " The theorem, as proved in this way, is seen to be true only when the point P lies on one of the four circles. To prove it in all generality we need only recall that Q is determinate so long as C and K are fixed and that we can replace any pair of orthogonal circles, such as O and O′, by another orthogonal pair belonging to the same coaxal system, without affecting the final point Q.

II

1. A conic may be regarded as the envelope of a variable tangent. Thus the inverse of a conic with respect to the origin O (which may be easily proved to be a nodal bicircular quartic, the common points of the variable circles being nodes on the curve) is the envelope of a variable circle passing through O. The locus of the centres of these circles is clearly a conic which is called the Focal Conic of the quartic.

2. *Theorem.*—If a system of co-axal circles intersect in two real points O and O′, and if through one of these O, two straight lines are drawn to cut the system at P, P′, P″, ..., Q, Q′, Q″,......, the envelope of the straight lines PQ, P′Q′, P″Q″, ... will be a parabola of which the other point O′ will be the focus.

For, since O′ is a point on every circle, the pedal line L of O′ with respect to the triangles OPQ, OP′Q′, ... will evidently be the same. Therefore, PQ, P′Q′, P″Q″, ... will all envelope the first negative pedal of L with respect to O′, which is a parabola having O′ for the focus and L the tangent at the vertex. Clearly, the parabola also touches each of the given straight lines.

3. Now invert the whole figure with respect to O′. Then we have the theorem : " If two circles cut in two real points O and O′,

and if through one of these O an infinity of straight lines POQ, P′OQ′, P″OQ″, ... be drawn, then the circumcircles of the triangles O′PQ, O′P′Q′, O′P″Q″,. .. ... envelop a cardioide touching the given circles.

Since when a straight line inverts into a circle, the image of the origin in that straight line inverts into the centre of that circle, we see that the centres of the circumscribing circles lie on another circle passing through the origin O′. This is, therefore, the Focal Circle of the cardioide. If the given circles are orthogonal, the focal circle will also pass through O.

Hence *the directrix of the parabola inverts into the focal circle of the cardioide.*

The above theorem may be stated in various ways; thus,—

"If the three angles of a triangle are given while the vertex is fixed and the base passes through a fixed point, the circumscribing circle of the triangle envelops a cardioide, and its centre describes a circle passing through the vertex and through the fixed point if the vertical angle be a right angle.

4. Next let us invert the theorem of § 2 with respect to O. Let us, moreover, suppose the given lines through O to be at right angles. We thus find the following theorem. "If a system of straight lines be drawn through a point O′ to cut a given pair of perpendicular lines through O, at the points PQ, P′Q′,......, then the envelope of the circumcircles OPQ, OP′Q′,......will be a bicircular quartic having a cusp at O."*

Further, it is seen that since O was on the directrix of the parabola, the focal conic of the quartic will be a rectangular hyperbola whose asymptotes are parallel to the given lines. Hence also, the elementary theorem: "The locus of the middle points of the segments intercepted between two given perpendicular straight lines of any number of straight lines drawn through a given point is a rectangular hyperbola passing through that point, of which the asymptotes are parallel to the given straight lines.

* The quartic will touch the given lines at their points of intersection with straight lines through O′ parallel to them.

III

1. The correspondence between successive inversions and rotations is established in the classical memoirs of (among others) Klein, Cayley and Poincare. We shall here only make a few remarks on the Cross Ratio Group of Projective Geometry,—

$$z, \ \frac{1}{z}, \ 1-z, \ \frac{1}{1-z}, \ \frac{z-1}{z}, \ \frac{z}{z-1}.$$

2. It is well-known that the non-homogeneous substitutions of the Dihedral Group are—

$$z'=e^{\frac{2ik\pi}{n}}z, \qquad z'=e^{\frac{2ik\pi}{n}}\frac{1}{z}$$

which are derivable from Cayley's formula [see, *e.g.*, Forsyth Theory of Functions, ch. xxii, Art. 300].

The cross ratio group in question is nothing other than the Dihedral group for $n=3$.

3. *Lemma.*—The direction of the axis being the same, if the origin be transferred to a point O′ (f, g, h) and if all the points of the points of the z—plane are centrally projected from $(0, 0, 1)$ to the new plane of z', to find the relation between the corresponding values of z, z'.

It can be seen without difficulty that the relation in question is

$$z'=(1-h)z-(f+ig).$$

This transformation will be real if $g=0$, when we have

$$z'=(1-h)z-f \qquad \dots \quad (a)$$

4. Now take the case of the Dihedron in which $n=3$.

The polygon here will therefore be an equilateral triangle. Take its plane to be the plane of $\eta=0$. Let the summits be called A, B, C,—where C, the point $(0, 0, 1)$ is also the vertex of projection.

We are now to choose the new origin O′. Since the cross ratio group permutes the numbers 0, 1, ∞ among themselves, it is clear that the transformation must be made in which the three summits correspond to the numbers 0, 1, ∞ on the axis of real numbers.

Hence, for instance, as the point A is to project to zero, the point O' must be in CA. Thus the origin must lie in the plane of $\eta=0$. And since the point B is to become by projection the point 1, the length $O'B'$ (B' being the point where CB cuts the new x-axis) must equal unity. Hence without difficulty, the coordinates of O' are seen to be $\left(-\frac{1}{2}, 0, 1-\sqrt{\frac{3}{2}}\right)$. Here since $f=0$, the transformation is necessarily real and equation (a) of Lemma gives

$$z'=\frac{\sqrt{3}.\,z+1}{2}, \text{ or } z=\frac{2z'-1}{\sqrt{3}}.$$

The rotations belonging to the dihedral group are:—

(i) the rotation about a line through O, the centre of the sphere, perpendicular to the plane of the paper, (a) through 120°, (b) through 240°, (c) through 360°,—this last giving the identical substitution;

(ii) the rotation through 180° about each of the secondary axes through A,B and C.

Using Cayley's formula and then making the transformation here indicated, the corresponding substitutions are seen to be in the following order.

$$z'=\frac{z-1}{z},\quad z'=\frac{1}{1-z} \text{ and } z'=z;$$

$$z'=\frac{z}{z-1},\quad z'=\frac{1}{z},\ \text{and } z'=1-z.$$

Thus we have obtained all the substitutions of this group, and the six anharmonic ratios of four points in a straight line thus have a correspondence with the rotations of a sphere, or with their equivalents, the successive inversions in circles. Interpretations of the harmonic and equianharmonic groups are now easy and at the same time interesting.

On the use of Ritz's method for finding the vibration-frequencies of heterogeneous strings and membranes.

BY

N. K. MAJUMDAR.

[*Read September 23rd, 1917.*]

CONTENTS:

Introduction.

1. The object of the paper is to show how reliable results about the vibration-frequencies of *heterogeneous* strings and membranes can be obtained by the use of a method, the germs of which are found in Lord Rayleigh's writings, and which was first clearly expounded by Ritz.*

2. It is believed that no previous writer has applied this method to determine the vibration-frequencies of *heterogeneous* strings and membranes, although the method has found applications to numerous other problems by many investigators, including Ritz himself, who considered the vibration of plates,† Prof. A. E. H. Love, who studied the theory of tides,‡ Prof. Kalahne and Dr. Reinstein.

* *Crelle's Journal*, Vol. CXXXV.

† *Annalen der Physik*, Vol. XXVIII, 1909.

‡ *Fifth International Congress of Mathematicians*, Vol. II, 1912.

Ritz's method.

3. If it is required to render the integral

$$J \equiv \int_{x_0}^{x_1} f_1(x, y, y', y'', \ldots y^{(\sigma)})\,dx$$

an extremum under the isoperimetric condition

$$I \equiv \int_{x_0}^{x_1} f_2(x, y, y', y'', \ldots y^{(\sigma)})\,dx = \text{constant},$$

the problem is equivalent to that of rendering $J+\lambda I$ an extremum without any isoperimetric condition.

It is well known that y must satisfy, as a necessary condition, some differential equation $D=0$, although every solution of $D=0$ may not render $J+\lambda I$ an extremum.

Conversely, if a solution of $D=0$ is required, which satisfies certain boundary conditions, and if we can obtain the corresponding isoperimetrical problem of the calculus of variations, the required solution of the differential equation may be taken any 'y' which renders $J+\lambda I$ an extremum.

Ritz's method consists in obtaining successive approximations to the value of y by the following process:—

Substitute for y in the integral $J+\lambda I \equiv J'$ say, $Y_n \equiv y_0 + a_1 y_1 + \ldots + a_n y_n$, where $y_0, y_1, y_2, \ldots y_n$ are known functions, $a_1, a_2, \ldots a_n$ are constants to be determined from the condition of rendering $J+\lambda I$ an extremum, and Y_n satisfies the prescribed boundary conditions.

By this substition J' becomes a known function $J_n(a_1, a_2, \ldots a_n)$ of the a's, independent of x. The a's are determined so that J_n may be an extremum, *i.e.*, from the n equations

$$\frac{\partial J_n}{\partial a_r} = 0, \ (r=1, 2, \ldots n) \qquad \ldots \qquad \ldots \quad \text{(A)}$$

In the boundary value problems of mathematical physics, we have to deal mostly with linear differential equations. J_n is thus a function of the second degree in the a's, and the equations (A) are therefore linear in the a's. There exists thus one and only one solution of the system (A), and we get the following successive approximations to the value of Y—

$$Y_1 = y_0 + a_1 y_1,$$

$$Y_2 = y_0 + a_1 y_1 + a_2 y_2,$$

etc., etc.

Heterogeneous Strings.

4. If $\rho =$ density, the equation of motion is:

$$\rho \frac{d^2 w}{dt^2} = \frac{d^2 w}{dx^2};$$

on putting $w = \cos kt \cdot y$, this reduces to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + k^2 \rho y = 0;$$

the boundary conditions being, say, $y(\pm 1) = 0$.

Any y will satisfy this differential equation, if it renders the integral

$$J' \equiv \int_{-1}^{+1} y'^2 \, dx$$

an extremum, and at the same time

$$I \equiv \int_{-1}^{+1} \rho y^2 \, dx = 1;$$

i.e., if it renders

$$J \equiv J' - k^2 I \equiv \int_{-1}^{+1} (y'^2 - k^2 \rho y^2)\, dx$$

an extremum without any isoperimetric condition.

4. CASE I: $\rho = 1 + qx^2$. *First* Approximation.

Put for y in J, $y_1 \equiv ((1-x^2)\ (a_0 + a_1 x^2)$, (which satisfies the necessary boundary conditions).

$$\text{Then}\quad J_1 = \int_{-1}^{+1} [y'^2 - k^2\ (1+qx^2) y^2] dx$$

$$= \int_{-1}^{+1} [\ \{\ (a_1 - a_0) - 2a_1 x^2\}^2\ (4x^2)$$

$$- k^2\ (1+qx^2)\ (1-x^2)^2\ (a_0 + a_1 x^2)^2] dx.$$

The system of equations, $\dfrac{\partial J_1}{\partial a_0} = 0$, $\dfrac{\partial J_1}{\partial a_1} = 0$, on the elimination of a_0 and a_1, leads to the following equation for the determination of k^2—

$$k^4 \left[1 + \frac{6}{11} q + \frac{1}{33} q^2\right] - k^2 \left[28 + \frac{48}{11} q\right] + 63 = 0,$$

which agrees with Ritz's equation,

$$k^4 - 28k^2 + 63 = 0,$$

for the case $q = 0$.

If q is considered to be very small, neglecting q^2 in the solution of the above equation, the first approximation to the fundamental-tone is given by the least root:

$$2k_1{}^2 = 4 \cdot 93\ (1 - q \times \cdot 134).$$

According to Lord Rayleigh's formula*

$$2k_1{}^2 = \frac{\pi^2}{2}(1-q\theta),$$

where

$$\theta = \int_0^2 (x-1)^2 \sin^2 \frac{\pi x}{2}\, dx = \cdot 131.$$

5. CASE I: $\rho = 1 + qx^2$. *Second* approximation.

In the definite integral, put

$$Y \equiv (1-x^2)\,(a_0 + a_1 x^2 + a_2 x^4),$$

when we get

$$J_2 \equiv \int_{-1}^{+1} [\{a_0 + a_1 (2x^2-1) + a_2 (3x^4 - 2x^2)\} (4x^2)$$

$$- k^2(1+qx^2)\,(1-x^2)^2\,(a_0 + a_1 x^2 + a_2 x^4)^2]dx.$$

The elimination of a_0, a_1, a_2 from the system

$$\frac{\partial J_2}{\partial a_r} = 0, \; (r = 0, 1, 2)$$

leads to the following equation for the determination of $\lambda = 2k^2$, neglecting q^2 and higher powers, *viz.*,

$$38610 - \lambda\,(8910 + 1422q) + \lambda^2\,(225 + 114q) - \lambda^3\,(1+q) = 0,$$

whence for the fundamental note we have

$$2k_1{}^2 = \lambda_1 = 4 \cdot 9348\,(1 - \cdot 1306q) \text{ approx.}$$

7. CASE II: $\rho = 1 + q \cos 2x$. First approximation.

Substituting in J

$$Y \equiv (1-x^2)\,(a_0 + a_1 x^2),$$

* Lord Rayleigh, *Theory of Sound*, Vol. I, p. 118.

we get

$$J_1 \equiv \int_{-1}^{+1} [\{a_0+a_1\,(2x^2-1)\}^2\,(4x^2)$$

$$-k^2\,(1+q\cos 2x)\,(1-x^2)^2\,(a_0+a_1\,x^2)^2]dx.$$

If $q\cos 2=8p$, the elimination of a_0 and a_1 from $\frac{\partial J_1}{\partial a_0}=0$ and $\frac{\partial J_1}{\partial a_1}=0$

leads to the following equation for $2k^2$:

$$k^4\left[16-210p\,(3549+\frac{3247}{2}t)\right]$$

$$-k^2\left[448-315p\,(5997+\frac{5471}{2}t)\right]+1008=0,$$

neglecting p^2 and higher powers, as q and hence p is supposed to be very small; also $t=\tan 2 \doteq -2\frac{1}{5}$ nearly. This equation again tallies with Ritz's equation

$$k^4-28k^2+63=0$$

for the case $q=0$.

The fundamental note, as a first approximation, is given by the least root

$$2k_1{}^2=4\cdot 93\,(1-\cdot 656q).$$

According to Lord Rayleigh,

$$2k_1{}^2=\frac{\pi^2}{2}\,(1-q\theta),$$

where

$$\theta=\frac{\pi^2\sin 2}{2(\pi^2 4)}\doteq\cdot 75 \text{ nearly}.$$

Heterogeneous square membrane.

8. The equation of motion is

$$\rho\frac{d^2w}{dt^2}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2};$$

putting

$$w = V \cdot \cos 2kt,$$

we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 4k^2 \rho V = 0,$$

where V satisfies some prescribed boundary conditions, say,

$$(V)_{x=\pm 1} = 0, \ (V)_{y=\pm 1} = 0.$$

Here we must render

$$J' \equiv \int_{-1}^{+1} \int_{-1}^{+1} \left\{ \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 \right\} dx\,dy$$

an extremum, under the isoperimetric condition

$$I \equiv \int_{-1}^{+1} \int_{-1}^{+1} \rho V^2 dx dy = \text{constant},$$

which is equivalent to rendering

$$J \equiv J' - k^2 I \equiv \int_{-1}^{+1} \int_{-1}^{+1} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 - 4k^2 \rho V^2 \right] dx dy$$

an extremum without any isoperimetric condition.

By Ritz's method $V(x, y)$ is to be found as successive approximations to

$$V_{m,n} \equiv (1-x^2)(1-y^2)[a_{0,0} + (a_{1,0}\, x^2 + a_{0,1}\, y^2) + (a_{2,0}\, x^4 + a_{1,1}\, x^2 y^2 + a_{0,2}\, y^4) + \ldots].$$

9. CASE I: $\rho = 1$. *Homogeneous* square membrane. First approximation.

As a first approximation, we substitute for V in J

$$V_{1,1} \equiv (1-x^2)(1-y^2)(a + bx^2 + cy^2).$$

The elimination of a, b, c from

$$\frac{\partial J_{1,1}}{\partial a} = 0, \ \frac{\partial J_{1,1}}{\partial b} = 0, \ \frac{\partial J_{1,1}}{\partial c} = 0,$$

leads to the following equation

$$(k^2-7)\,(32k^4-264k^2+277)=0,$$

the roots of which are

$$k_1{}^2=1\cdot 234,$$

$$k_2{}^2=7,$$

$$k_3{}^2=7\cdot 01, \text{ nearly.}$$

10. CASE II : $\rho=1+qx^2y^2$ (*Heterogeneous*): First approximation.

Substituting $V_{1,1}$ in J as before, and eliminating a, b, c from the equations

$$\frac{\partial J_{1,1}}{\partial a}=\frac{\partial J_{1,1}}{\partial b}=\frac{\partial J_{1,1}}{\partial c}=0,$$

we get the following equation for k^2 ;

$$\left[7-k^2\left(1+\frac{q}{3\times 7\times 11}\right)\right]+\left[k^4\left\{32-\frac{1058\times 16}{11\times(21)^2}\,q-\frac{16\times 680}{(21)^5\times 11}\,q^2\right\}\right.$$

$$\left.-k^2\left\{264-\frac{4\times 3332}{(21)^2\times 11}\,q\right\}+277\right]=0.$$

Considering q to be small, and neglecting q^2 and higher powers, the roots are

$$k_1{}^2=1\cdot 234(1-q\times\cdot 008),$$

$$k_2{}^2=7(1-q\times\cdot 004),$$

$$k_3{}^2=7\cdot 01(1+q\times\cdot 11).$$

Before I conclude I must express my deep sense of gratitude and indebtedness to Dr. Ganesh Prasad who kindly suggested to me the problem and helped me with other directions.

On the steady motion of a viscous fluid due to the rotation of two rigid bodies about arbitrary axes

BY

BIJON DUTT.

[*Read March 9th, 1919.*]

The present paper is the first instalment of the results of my investigation on the mutual influence between any two given bodies capable of rotating about any two given axes in a viscous fluid medium. The simplest case of this problem, *viz.*, that in which the given bodies are spheres capable of rotating about the line through their centres, has been recently studied by Mr. G. B. Jeffery.*

In the present paper, I have given the complete solution for two more cases, *viz.*, the case of two spheroids rotating about a common axis and two cylinders rotating about two parallel axes. Other cases are also being studied by me and these will be given in a subsequent paper.

I wish to express my obligation to Dr. S. K. Banerji at whose suggestion I took up and under whose guidance I carried on the investigation.

Case I. *Two Spheroids rotating about a Common Axis.*

Suppose that a point P has the polar coordinates (r,θ) and (r',θ') referred to two points O and O′, θ and θ' being measured in opposite senses from the line OO′, and let OO′$=c$.

* "On the steady motion of a solid of revolution in a viscous fluid." (*Proceedings of the London Mathematical Society*, February, 1915).

Then the following transformation theorems are well-known :—

$$\text{(I).}\quad \frac{P'^m_n}{r'^{n+1}} = \frac{r^m}{(n-m)!\;c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P^m_m\right.$$

$$\left. + \frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P^m_{m+1} + \dots + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P^m_{m+s} + \dots\right]$$

(if $r < c$),

$$\text{(II).}\quad \frac{P^m_n}{r^{n+1}} = \frac{r'^m}{(n-m)!\;c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P'^m_m\right.$$

$$\left. + \frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P'^m_{m+1} + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P'^m_{m+s} + \dots\right]$$

(if $r' < c$),

$P^m_n(\cos\theta)$ being an associated Legendre function of degree n and order m whose origin is O and the axis is OO′ and $P'^m_n(\cos\theta')$ a similar function whose origin is O′ and the axis is O′O.

The equations of the spheroids with centres at O and O′ and with OO′ as the common axis of revolution, can be written as

$$r=a\,[1+\epsilon\, P_2\,(\cos\theta)], \qquad \dots \qquad \dots \quad (1)$$

$$r'=b\,[1+\epsilon'\, P'_2\,(\cos\theta')], \qquad \dots \qquad \dots \quad (2)$$

where ϵ and ϵ' are two small quantities whose second and higher powers are supposed to be negligible.

Let ω and ω' denote the angular velocities of the spheroids.

Mr. Jeffery has shown* that if (ρ, z, ϕ) are a set of cylindrical coordinates and if we have a solution of Laplace's equation of the form

$$f(\rho, z)\sin\phi, \qquad \dots \qquad \dots \quad (3)$$

then the solution of a slow, steady and symmetrical motion about the axis of z is given by

$$v=f(\rho, z), \qquad \dots \qquad \dots \quad (4)$$

where v is the velocity at any point of the fluid.

* *Proceedings of the London Mathematical Society*, February, 1915.

Now it is well known that

$$r^{-n-1} P^1_n (\cos\theta) \sin\phi \quad \dots \qquad \dots \quad (5)$$

is a solution of Laplace's Equation.

We accordingly assume as the solution of our present problem

$$v = \sum_{n=1}^{\infty} \left[\frac{A_n}{r^{n+1}} P^1_n(\cos\theta) + \frac{B_n}{r'^{n+1}} P'^1_n(\cos\theta') \right], \quad \dots \quad (6)$$

A_n, B_n being arbitrary constants to be determined.

This expression for v apparently vanishes at infinity. It remains to determine the constants so that the following two conditions may also be satisfied

$$v = \omega r \sin\theta, \qquad \text{when } r = a[1 + \epsilon P_2 (\cos\theta)], \quad \dots \quad (7)$$

$$v = \omega' r' \sin\theta', \qquad \text{when } r' = b[1 + \epsilon' P'_2 (\cos\theta')]. \quad \dots \quad (8)$$

Putting $m=1$ in "Theorem I," we get

$$\frac{P'^1_n}{r'^{n+1}} = \frac{r}{(n-1)!\, c^{n+2}} \left[\frac{(n+1)!}{2!} P_1^1 + \frac{(n+2)!}{3!} \frac{r}{c} P_2^1 + \dots \right.$$

$$\left. + \frac{(n+s+1)!}{(s+2)!} \left(\frac{r}{c}\right)^s P^1_{s+1} + \dots \right]. \quad \dots \quad (9)$$

Therefore

$$v = \sum_{n=1}^{\infty} \frac{A_n}{r^{n+1}} P_n^1 (\cos\theta) + \sum_{k=1}^{\infty} \frac{R_{k+1}}{k+1} \frac{r^k}{c^{k+1}} P_k^1 (\cos\theta), \dots (10)$$

where

$$R_{k+1} = (k+1) \frac{B_1}{c} + \frac{(k+1)(k+2)}{1!} \frac{B_2}{c^2} + \frac{(k+1)(k+2)(k+3)}{2!} \frac{B_3}{c^3}$$

$$+ \dots \qquad \dots (11)$$

We note the following well known properties of Legendre's co-efficients :—

$$P_n^1(\mu)=(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu}, \quad \ldots \quad \ldots \quad \ldots (12)$$

$$\frac{dP_n(\mu)}{d\mu}=(2n-1)\, P_{n-1}(\mu)+(2n-5)\, P_{n-3}(\mu)$$
$$+(2n-9)\, P_{n-5}(\mu)+\ldots, \quad \ldots (13)$$

the last term being $P_0(\mu)$ or $3P_1(\mu)$ according as n is odd or even

$$(\mu^2-1)\frac{dP_n(\mu)}{d\mu}=n\mu P_n(\mu)-nP_{n-1}(\mu), \quad \ldots \quad \ldots (14)$$

$$nP_n(\mu)-(2n-1)\,\mu P_{n-1}(\mu)+(n-1)\, P_{n-2}(\mu)=0 \quad \ldots (15)$$

Hence

$$P_2(\mu)P_n^1(\mu)=\frac{3\mu^2-1}{2}(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu} \quad \text{[by (12)}$$

$$=\left\{\frac{3}{2}(\mu^2-1)+1\right\}(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu}$$

$$=(1-\mu^2)^{\frac{1}{2}}\left[\frac{3}{2}\{n\mu P_n(\mu)-nP_{n-1}(\mu)\}\right.$$
$$\left.+\{(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\}\right]$$

[by (13), (14).

Therefore using (15), we have

$$P_2(\mu)P_n^1(\mu)=(1-\mu^2)^{\frac{1}{2}}\left[\frac{3}{2}\,\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu)-P_{n-1}(\mu)\}\right.$$
$$\left.+\{(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\}\right]\ldots (16)$$

Now

$$r=a[1+\epsilon P_2(\cos\theta)],$$

Therefore neglecting the second and higher powers of ϵ,

$$r^k = a^k[1 + k\epsilon P_2(\cos\theta)], \quad \ldots \quad \ldots (17)$$

$$\frac{1}{r^{n+1}} = \frac{1}{a^{n+1}}[1 - (n+1)\epsilon P_2(\cos\theta)]. \quad \ldots (18)$$

Therefore on writing μ for $\cos\theta$,

$$r^k P_k{}^1(\cos\theta) = a^k[1 + k\epsilon P_2(\mu)]P_k{}^1(\mu);$$

or,

$$r^k P_k{}^1(\cos\theta) = a^k\Big[\{(2n-1)P_{k-1}(\mu) + (2n-5)P_{k-3}(\mu) + \ldots\}$$

$$+ \epsilon k\Big\{\frac{3}{2}\,\frac{k(k+1)}{2k+1}\{P_{k+1}(\mu) - P_{k-1}(\mu)\} + (2k-1)P_{k-1}(\mu)$$

$$+ (2k-5)P_{k-3}(\mu) + \ldots\Big\}\Big] \quad \ldots (19)$$

and

$$\frac{P_n{}^1(\cos\theta)}{r^{n+1}} = \frac{1}{a^{n+1}}\{1 - (n+1)\epsilon P_2(\mu)\}P_n{}^1(\mu)$$

or,

$$\frac{P_n{}^1(\cos\theta)}{r^{n+1}} = \frac{1}{a^{n+1}}(1-\mu^2)^{\frac{1}{2}}\Big[\{(2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu)$$

$$+ \ldots\} - \epsilon(n+1)\Big\{\frac{3}{2}\,\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu) - P_{n-1}(\mu)\}$$

$$+ (2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu) + \ldots\Big\}\Big]. \quad \ldots (20)$$

Also

$$\omega r \sin\theta = wa[1 + \epsilon P_2(\cos\theta)](1 - \cos^2\theta)^{\frac{1}{2}}$$

or,

$$\omega r \sin\theta = wa(1-\mu^2)^{\frac{1}{2}}[1 + \epsilon P_2(\mu)] \quad \ldots \quad \ldots (21)$$

Therefore the condition (7) is equivalent to

$$\omega a^2[1+\epsilon P_2(\mu)]=\sum_{n=1}^{\infty}\frac{A_n}{a^n}\Big[\{(2n-1)\,P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\}$$

$$-\epsilon\,(n+1)\left[\frac{3}{2}\,\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu)-P_{n-1}(\mu)\}+(2n-1)\,P_{n-1}(\mu)+\ldots\}\right]$$

$$+\sum_{k=1}^{\infty}\frac{R_{k+1}}{k+1}\left(\frac{a}{c}\right)^{k+1}\Big[\{(2k-1)P_{k-1}(\mu)+(2k-5)P_{k-3}(\mu)+\ \ldots\}$$

$$+\epsilon k\left\{\frac{3}{2}\,\frac{k(k+1)}{2k+1}\{P_{k+1}(\mu)-P_{k-1}(\mu)\}+(2k-1)P_{k-1}(\mu)+\ldots\right\}\Big]\quad(22)$$

Equating the co-efficients of $P_0(\mu)$ from both sides of this identity, we get

$$\omega a^2=\left(\frac{A_1}{a}+\frac{A_3}{a^3}+\frac{A_5}{a^5}+\ldots\right)-\epsilon\left(\frac{4A_3}{a^3}+\frac{6A_5}{a^5}+\ldots\right)$$

$$+\left(\frac{a^2}{c^2}\,\frac{R_2}{2}+\frac{a^4}{c^4}\,\frac{R_4}{4}+\ldots\right)+\epsilon\left(\frac{3a^4}{c^4}\,\frac{R_4}{4}+\frac{5a^6}{c^6}\,\frac{R_6}{6}+\ldots\right)\quad(23)$$

Equating the co-efficients of $P_2(\mu)$ similarly, we get

$$\omega a^2\epsilon=5\left(\frac{A_3}{a^3}+\frac{A_5}{a^5}+\ldots\right)-5\epsilon\left(\frac{4A_3}{a^3}+\frac{6A_5}{a^5}+\ldots\right)$$

$$+5\left(\frac{a^4}{c^4}\,\frac{R_4}{4}+\frac{a^6}{c^6}\,\frac{R_6}{6}+\ldots\right)+5\epsilon\left(\frac{3a^4}{c^4}\,\frac{R_4}{4}+\frac{5a^6}{c^6}\,\frac{R_6}{6}+\ldots\right)$$

$$-\epsilon\,\frac{2A_1}{a}+\epsilon\,\frac{72}{7}\,\frac{A_3}{a^3}+\epsilon\frac{a^2}{c^2}\,\frac{R_2}{2}-\epsilon\,\frac{27}{14}\,\frac{a^4}{c^4}\,R_4\qquad\ldots\qquad\ldots\quad(24)$$

Hence by subtraction,

$$\omega a^2\,(5-\epsilon)=\frac{A_1}{a}\,(5+2\epsilon)-\epsilon\,\frac{72}{7}\,\frac{A_3}{a^4}+\frac{a^2}{2c^2}\,(5-\epsilon)\,R_2+\epsilon\,\frac{27}{14}\,\frac{a^4}{c^4}\,\frac{R_4}{4}\quad(25)$$

Neglecting terms of orders higher than that of $\frac{a^2}{c^2}$ this becomes

$$\omega a^2 (5-\epsilon)=\frac{A_1}{a}(5+2\epsilon) \quad \dots \quad \dots \quad \dots \quad \dots \quad (26)$$

$$\therefore \quad A_1=\omega a^3 \left(\frac{5-\epsilon}{5+2\epsilon}\right) \quad \dots \quad \dots \quad \dots \quad \dots \quad (27)$$

Consequently by symmetry,

$$B_1=\omega' b \left(\frac{5-\epsilon'}{5+2\epsilon'}\right) \quad \dots \quad \dots \quad \dots \quad \dots \quad (28)$$

Thus to this order of approximation

$$v=\omega a^3 \left(\frac{5-\epsilon}{5+2\epsilon}\right)\frac{P_1{}^1(\cos\theta)}{r^2}+\omega' b^3 \left(\frac{5-\epsilon'}{5+2\epsilon'}\right)\frac{P'_1{}^1(\cos\theta')}{r'^2} \dots \quad (29)$$

For higher approximations we proceed as follows :—

Equating the co-efficients of $P_n(\mu)$ from both sides of the identity (22), we get for $n>0$ and $\neq 2$

$$0=(2n+1)\left(\frac{A_{n+1}}{a^{n+1}}+\frac{A_{n+3}}{a^{n+3}}+\dots\right)-\epsilon(2n+1)\left(\frac{n+2}{a^{n+1}}A_{n+1}+\frac{n+4}{a^{n+3}}A_{n+3}+\dots\right)$$

$$+(2n+1)\left\{\left(\frac{a}{c}\right)^{n+2}\frac{R_{n+2}}{n+2}+\left(\frac{a}{c}\right)^{n+4}\frac{R_{n+4}}{n+4}+\dots\right\}$$

$$+\epsilon(2n+1)\left\{\frac{n+1}{n+2}\left(\frac{a}{c}\right)^{n+2}R_{n+2}+\frac{n+3}{n+4}\left(\frac{a}{c}\right)^{n+4}R_{n+4}+\dots\right\}$$

$$-\epsilon\frac{3n^2(n-1)}{2(2n-1)}\frac{A_{n-1}}{a^{n-1}}+\epsilon\frac{3(n+1)(n+2)^2}{2(2n+3)}\frac{A_{n+1}}{a^{n+1}}$$

$$+\epsilon\frac{3(n-1)^2}{2(2n-1)}\left(\frac{a}{c}\right)^{n}R_n-\epsilon\frac{3(n+1)^2}{2(2n+3)}\left(\frac{a}{c}\right)^{n+2}R_{n+2} \quad \dots \quad (30)$$

Similarly equating the co-efficients of $P_{n+2}(\mu)$, we get

$$0=(2n+5)\left(\frac{A_{n+3}}{a^{n+3}}+\frac{A_{n+5}}{a^{n+5}}+\ldots\right)-\epsilon(2n+5)\left(\frac{n+4}{a^{n+3}}A_{n+3}+\frac{n+6}{a^{n+5}}A_{n+5}+\ldots\right)$$

$$+(2n+5)\left\{\left(\frac{a}{c}\right)^{n+4}\frac{R_{n+4}}{n+4}+\left(\frac{a}{c}\right)^{n+6}\frac{R_{n+6}}{n+6}+\ldots\right\}$$

$$+\epsilon(2n+5)\left\{\frac{n+3}{n+4}\left(\frac{a}{c}\right)^{n+4}R_{n+4}+\frac{n+5}{n+6}\left(\frac{a}{c}\right)^{n+6}R_{n+6}+\ldots\right\}$$

$$-\epsilon\frac{3(n+2)^2(n+1)}{2(2n+3)}\frac{A_{n+1}}{a^{n+1}}+\epsilon\frac{3(n+3)(n+4)^2}{2(2n+7)}\frac{A_{n+3}}{a^{n+3}}$$

$$+\epsilon\frac{3(n+1)^2}{2(2n+3)}\left(\frac{a}{c}\right)^{n+2}R_{n+2}-\epsilon\frac{3(n+3)^2}{2(2n+7)}\left(\frac{a}{c}\right)^{n+4}R_{n+4} \qquad (31)$$

Multiplying (30) by $(2n+5)$ and (31) by $(2n+1)$ and subtracting we get

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$-\epsilon\frac{A_{n-1}}{a^{n-1}}\frac{3n^2(n-1)(2n+5)}{2(2n-1)}-\epsilon\frac{A_{n+3}}{a^{n+3}}\frac{3(n+3)(n+4)^2(2n+1)}{2(2n+7)}$$

$$+\frac{R_{n+2}}{n+2}\left(\frac{a}{c}\right)^{n+2}\left[(2n+1)(2n+5)+\epsilon(n^2+3n-1)(n+1)\right]$$

$$+\epsilon R_n\left(\frac{a}{c}\right)^n\frac{3(n-1)^2(2n+5)}{2(2n-1)}+\epsilon\left(\frac{a}{c}\right)^{n+4}R_{n+4}\frac{3(n+3)^2(2n+1)}{2(2n+7)} \qquad (32)$$

Neglecting the terms containing A_{n+3} and R_{n+4} we have

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$-\epsilon\frac{A_{n-1}}{a^{n-1}}\frac{3n^2(n-1)(2n+5)}{2(2n-1)}$$

$$+\frac{R_{n+2}}{n+2}\left(\frac{a}{c}\right)^{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon\right]$$

$$+\epsilon\left(\frac{a}{c}\right)^n R_n\frac{3(n-1)^2(2n+5)}{2(2n-1)} \qquad \ldots \qquad (33)$$

Putting $n-2$ for n in this equation, we get

$$0=\frac{A_{n-1}}{a^{n-1}}\left[(2n-3)(2n+1)-n(n^2-n-3)\,\epsilon\right]$$

$$-\epsilon\,\frac{A_{n-3}}{a^{n-3}}\,\frac{3(n-2)^2(n-3)(2n+1)}{2n-5}$$

$$+\frac{a^n}{c^n}\,\frac{R_n}{n}\left[(2n-3)(2n+1)+(n+1)(n^2-n-3)\epsilon\right]$$

$$+\frac{a^{n-2}}{c^{n-2}}\,\epsilon\,R_{n-2}\,\frac{3(n-3)^2(2n+1)}{2(2n-5)} \qquad \ldots \qquad (34)$$

Substituting the value of A_{n-1} from (34) in (33) and neglecting ϵ^2, we have

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$+\left(\frac{a}{c}\right)^{n+2}\frac{R_{n+2}}{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon\right]$$

$$+\left(\frac{a}{c}\right)^{n}R_n\,\epsilon\,\frac{3}{2}(n-1)(2n+5) \qquad \ldots \qquad \ldots \qquad (35)$$

By symmetry

$$0=\frac{B_{n+1}}{b^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon'\right]$$

$$+\left(\frac{b}{c}\right)^{n+2}\frac{S_{n+2}}{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon'\right]$$

$$+\left(\frac{b}{c}\right)^{n}S_n\,\epsilon'\,\frac{3}{2}(n-1)(2n+5), \qquad \ldots \qquad \ldots \qquad (36)$$

where

$$S_{k+1}=(k+1)\,\frac{A_1}{c}+\frac{(k+1)(k+2)}{1!}\,\frac{A_2}{c^2}$$

$$+\frac{(k+1)(k+2)(k+3)}{2!}\,\frac{A_3}{c_3}+\ldots \qquad \ldots \qquad (37)$$

Therefore, for $n > 0$ and $\neq 2$,

$$A_{n+1} = -\frac{a^{2n+3}}{c^{n+2}}\frac{R_{n+2}}{n+2}\left[1+\frac{(n^2+3n-1)(2n+3)\epsilon}{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon}\right]$$

$$-\frac{a^{2n+1}}{c^n}R_n\epsilon\frac{3(n-1)(2n+5)}{2\{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\}} \quad (38)$$

$$B_{n+1} = -\frac{b^{2n+3}}{c^{n+2}}\frac{S_{n+2}}{n+2}\left[1+\frac{(n^2+3n-1)(2n+3)\epsilon'}{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon'}\right]$$

$$-\frac{b^{2n+1}}{c^n}S_n\epsilon'\frac{3(n-1)(2n+5)}{2\{(2n+1)(2n+5)-(n^2+3n-5)(n+2)\epsilon'\}} \quad (39)$$

To determine A_3 and B_3, we proceed as follows :—

On equating the co-efficients of $P_4(\mu)$ from both sides of (22), we have

$$0 = 9\left[\frac{A_5}{a^5}+\frac{A_7}{a^7}+\dots\right]-9\epsilon\left[6\frac{A_5}{a^5}+8\frac{A_7}{a^7}+\dots\right]$$

$$+9\left[\frac{a^6}{c^6}\frac{R_6}{6}+\frac{a^8}{c^8}\frac{R_8}{8}+\dots\right]+9\epsilon\left[\frac{5}{6}\frac{a^6}{c^6}R_6+\frac{7}{8}\frac{a^8}{c^8}R_8+\dots\right]$$

$$-\epsilon\frac{72}{7}\frac{A_3}{a^3}+\epsilon\frac{273}{11}\frac{A_5}{a^5}+\epsilon\frac{27}{14}\frac{a^4}{c^4}R_4-\epsilon\frac{75}{22}\frac{a^6}{c^6}R_6 \quad \dots \quad (40)$$

From (40) and (24), we get

$$9\omega a^2\epsilon = 9\frac{A_3}{a^3}(5-8\epsilon)-\epsilon\frac{18A_1}{a}+\frac{a^4}{c^4}R_4\frac{9}{4}(5-3\epsilon)+\epsilon\frac{9}{2}\frac{a^2}{c^2}R_2$$

$$+\epsilon\frac{325}{22}\frac{a^6}{c^6}R_6-\epsilon\frac{1350}{11}\frac{A_5}{a^5} \quad \dots \quad (41)$$

Neglecting the terms containing R_4, R_6 and A_5, we have

$$9\omega a^2\epsilon = 9\frac{A_3}{a^3}(5-8\epsilon)-\epsilon\frac{18A_1}{a}+\frac{9}{2}\epsilon\frac{a^2}{c^2}R_2 \quad \dots \quad \dots \quad (42)$$

Now

$$A_1 = \omega a^3 \left(\frac{5-\epsilon}{5+2\epsilon}\right), \ R_2 = 2\ \frac{\omega' b^3}{c} \left(\frac{5-\epsilon'}{5+2\epsilon'}\right)$$

Therefore

$$A_3 = 3\omega a^5 \left(\frac{\epsilon}{5-8\epsilon}\right) - \frac{\omega' a^5 b^3}{c^3} \left(\frac{5-\epsilon'}{5+2\epsilon'}\right) \frac{\epsilon}{5-8\epsilon} \qquad \ldots \quad (43)$$

Similary

$$B_3 = 3\omega' b^5 \left(\frac{\epsilon'}{5-8\epsilon'}\right) - \frac{\omega a^3 b^5}{c^3} \left(\frac{5-\epsilon}{5+2\epsilon}\right) \left(\frac{\epsilon'}{5-8\epsilon'}\right) \qquad \ldots \quad (44)$$

On calculating A_2 and B_2 by means of formula (38) and (39) we find

$$A_2 = -\omega' \frac{a^5 b^3}{c^4} \left(\frac{5-\epsilon'}{5+2\epsilon'}\right) \left\{ 1 + \frac{5\epsilon}{7-3\epsilon} \right\}$$

$$B_2 = -\omega \frac{a^3 b^5}{c^4} \left(\frac{5-\epsilon}{5+2\epsilon}\right) \left\{ 1 + \frac{5\epsilon'}{7-3\epsilon'} \right\}$$

Case II. *Two Circular Cylinders rotating about Parallel Axes.*

We will suppose the fluid to be incompressible and the external forces to be derivable from a potential V.

Then writing

$$\chi = -V - \frac{p}{\rho} \qquad \ldots \quad (1)$$

the equations of motion in two dimensions are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{\partial \chi}{\partial x} + \nu\nabla^2 u, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \frac{\partial \chi}{\partial y} + \nu\nabla^2 v, \end{aligned}\right\} \qquad \ldots \quad (2)$$

where ν stands for the kinematic coefficient of viscosity. Putting $\frac{\partial}{\partial t}=0$ for steady motion and neglecting the squares and products of velocity for slow motion, these equations reduce to

$$\left.\begin{aligned}\nabla^2 u=-\frac{1}{\nu}\frac{\partial\chi}{\partial x}\\ \nabla^2 v=-\frac{1}{\nu}\frac{\partial\chi}{\partial y}\end{aligned}\right\} \quad \dots \quad (3)$$

On eliminating χ we get

$$\nabla^2\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=0. \quad \dots \quad (4)$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \dots \quad (5)$$

Consequently there exists a function ψ of x and y, such that

$$u=-\frac{\partial\psi}{\partial y},\ v=\frac{\partial\psi}{\partial x} \quad \dots \quad (6)$$

Hence (4) becomes

$$\nabla^4\psi=0. \quad \dots \quad (7)$$

We take co-ordinates defined by

$$x+iy=c\tan\frac{1}{2}(\xi+i\eta). \quad \dots \quad (8)$$

Whence

$$x=\frac{c\sin\xi}{\cos\xi+\cosh\eta},\ y=\frac{c\sinh\eta}{\cos\xi+\cosh\eta}, \quad \dots \quad (9)$$

and

$$x^2+y^2-2cy\coth\eta+c^2=0 \quad \dots \quad (10)$$

Equation (10) represents two families of circles whose centres lie on the axis of y at distances $\pm$ c coth η from the origin, and whose radii are equal to c cosech η. These circles do not cut the axis of x.

We can choose the axes of reference and the constant c, so that any two given non-intersecting circles are members of these families.

We assume the rotating cylinders to be given by

$$\eta = \alpha, \qquad \ldots \qquad \ldots \qquad \ldots \quad (11)$$

$$\eta = -\beta. \qquad \ldots \qquad \ldots \qquad \ldots \quad (12)$$

We have

$$\nabla^2 = \frac{1}{\mathrm{AB}} \left[\frac{\partial}{\partial \xi}\left(\frac{\mathrm{B}}{\mathrm{A}} \frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\mathrm{A}}{\mathrm{B}} \frac{\partial}{\partial \eta}\right) \right], \quad \ldots \quad (13)$$

where

$$ds^2 = \mathrm{A}^2 d\xi^2 + \mathrm{B}^2 d\eta^2. \quad \ldots \qquad \ldots \qquad \ldots \quad (14)$$

From (9), we find

$$ds^2 = \frac{c^2}{(\cos \xi + \cosh \eta)^2} (d\xi^2 + d\eta^2), \qquad \ldots \quad (15)$$

so that

$$\mathrm{A} = \mathrm{B} = \frac{c}{\cos \xi + \cosh \eta} \qquad \ldots \quad (16)$$

Hence (13) reduces to

$$\nabla^2 = \frac{1}{c^2} (\cos \xi + \cosh \eta)^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \qquad \ldots \quad (17)$$

Writing

$$\nabla_1^2 \text{ for } \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2},$$

the equation (7) takes the form

$$\nabla_1^2[(\cos\xi+\cosh\eta)^2\nabla_1^2\psi]=0 \quad \dots \quad \dots \quad (18)$$

To solve this equation we shall write

$$\psi=\phi\theta \quad \dots \quad \dots \quad \dots \quad (19)$$

and assume

$$\nabla_1^2\phi=0. \quad \dots \quad \dots \quad \dots \quad (20)$$

Then (19) becomes

$$\nabla_1^2[(\cos\xi+\cosh\eta)^2\nabla_1^2(\phi\theta)]=0 \quad \dots \quad (21)$$

But

$$\nabla_1^2(\phi\theta)=\theta\nabla_1^2\phi+2\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right)+\phi\nabla_1^2\theta \quad \dots \quad (22)$$

Remembering that $\nabla_1^2\phi=0$, (21) then becomes

$$\nabla_1^2\left[(\cos\xi+\cosh\eta)^2\left\{2\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right)+\phi\nabla_1^2\theta\right\}\right] \dots \quad (23)$$

First, we assume

$$\theta=\frac{\cos\xi}{\cos\xi+\cosh\eta},$$

so that

$$\frac{\partial\theta}{\partial\xi}=\frac{-\sin\xi\cosh\eta}{(\cos\xi+\cosh\eta)^2},\quad \frac{\partial\theta}{\partial\eta}=\frac{-\cos\xi\sinh\eta}{(\cos\xi+\cosh\eta)^2},$$

$$\nabla_1^2\theta=-\frac{2}{(\cos\xi+\cosh\eta)^2}$$

and (23) becomes

$$\nabla_1^2\left[-2\sin\xi\cosh\eta\frac{\partial\phi}{\partial\xi}-2\cos\xi\sinh\eta\frac{\partial\phi}{\partial\eta}-2\phi\right]=0. \quad (24)$$

But

$$\nabla_1^2\left(\sin\xi\cosh\eta\frac{\partial\phi}{\partial\xi}\right)=\frac{\partial\phi}{\partial\eta}\nabla_1^2(\sin\xi\cosh\eta)$$

$$+2\frac{\partial}{\partial\xi}(\sin\xi\cosh\eta)\frac{\partial^2\phi}{\partial\xi^2}+2\frac{\partial}{\partial\eta}(\sin\xi\cosh\eta)\frac{\partial^2\phi}{\partial\xi\partial\eta}$$

$$+\sin\xi\cosh\eta\,\nabla_1^2\frac{\partial\phi}{\partial\xi}$$

$$=2\cos\xi\cosh\eta\frac{\partial^2\phi}{\partial\xi^2}+2\sin\xi\sinh\eta\frac{\partial^2\phi}{\partial\xi\partial\eta}.$$

$$\nabla_1^2\left(\cos\xi\sinh\eta\frac{\partial\phi}{\partial\eta}\right)=+2\cos\xi\cosh\eta\frac{\partial^2\phi}{\partial\eta^2}$$

$$-2\sin\xi\sinh\eta\frac{\partial^2\phi}{\partial\xi\partial\eta}.$$

Therefore remembering that $\nabla_1^2\phi=0$, the left hand side of (24) becomes

$$-4\cos\xi\cosh\eta\left(\frac{\partial^2\phi}{\partial\eta^2}+\frac{\partial^2\phi}{\partial\eta^2}\right)$$

which is zero since $\nabla_1^2\phi=0$.

Thus (23) is satisfied by this assumption for θ. Accordingly

$$\frac{\cos\xi}{\cos\xi+\cosh\eta}\phi$$

is a solution of (18).

We can similarly show that

$$\frac{\sin\xi}{\cos\xi+\cosh\eta}\phi,\ \frac{\cosh\eta}{\cos\xi+\cosh\eta}\phi \text{ and } \frac{\sinh\eta}{\cos\xi+\cosh\eta}\phi$$

are all solutions of (18).

Now

$$\begin{array}{c}\cos\\ \sin\end{array} n\xi \cdot \begin{array}{c}\cosh\\ \sinh\end{array} n\eta$$

are well known solutions of $\nabla_1^2\phi=0$.

Hence

$$\frac{\begin{array}{c}\cos\\ \sin\end{array}(n+1)\xi \cdot \begin{array}{c}\cosh\\ \sinh\end{array} n\eta}{\cos\xi+\cosh\eta}$$

and

$$\frac{\begin{array}{c}\cos\\ \sin\end{array} n\xi \begin{array}{c}\cosh\\ \sinh\end{array}(n+1)\eta}{\cos\xi+\cosh\eta}$$

are solutions of (18).

We accordingly make the following assumption for ψ.

$$\psi=\frac{1}{\cos\xi+\cosh\eta}\sum_{n=0}^{\infty}\sin(n+\tfrac{1}{2})\xi\left[A_n\frac{\sinh(n-\frac{1}{2})(\eta-\alpha)}{\sinh(n-\frac{1}{2})(\alpha+\beta)}\right.$$

$$+B_n\frac{\sinh(n+\frac{3}{2})(\eta-\alpha)}{\sinh(n+\frac{3}{2})(\alpha+\beta)}+C_n\frac{\cosh(n-\frac{1}{2})(\eta-\alpha)}{\cosh(n-\frac{1}{2})(\alpha+\beta)}$$

$$\left.+D_n\frac{\cosh(n+\frac{3}{2})(\eta-\alpha)}{\cosh(n+\frac{3}{2})(\alpha+\beta)}\right], \quad \ldots \quad (25)$$

where A_n, B_n, C_n, D_n are arbitrary constants which are to be determined so that the boundary conditions may be satisfied.

The velocity component tangential to the curve ξ=constant is

$$\frac{\partial\psi}{A\partial\xi} \text{ or } \frac{\cos\xi+\cosh\eta}{c}\frac{\partial\psi}{\partial\xi},$$

and that tangential to the curve η=constant is

$$\frac{\partial\psi}{B\partial\eta} \text{ or } \frac{\cos\xi+\cosh\eta}{c}\frac{\partial\psi}{\partial\eta}.$$

Therefore on the hypothesis of no slipping, the conditions to be satisfied on the surfaces of the cylinders are

$$(i) \quad \frac{\partial \psi}{\partial \xi}=0, \text{ when } \eta=a,$$

$$(ii) \quad \frac{\partial \psi}{\partial \xi}=0, \text{ when } \eta=\beta,$$

$$(iii) \quad \frac{\cos \xi+\cosh \eta}{c} \quad \frac{\partial \psi}{\partial \eta}=\omega a, \text{ when } \eta=a,$$

$$(iv) \quad \frac{\cos \xi+\cosh \eta}{c} \quad \frac{\partial \psi}{\partial \eta}=\omega' b, \text{ when } \eta=-\beta,$$

where ω, ω' are the angular velocities and a, b the radii of the cylinders.

The coefficient of $\sin (n+\frac{1}{2})\xi$ in (25) reduces to

$$\frac{C_n}{\cosh(n-\frac{1}{2})(a+\beta)}+\frac{D_n}{\cosh(n+\frac{3}{2})(a+\beta)} \text{ and } -A_n-B_n+C_n+D_n \text{ when}$$

we put $\eta=a$ and $\eta=-\beta$ respectively.

We make

$$\frac{C_n}{\cosh(n-\frac{1}{2})(a+\beta)}+\frac{D_n}{\cosh(n+\frac{3}{2})(a+\beta)}=0 \text{ and } -A_n-B_n+C_n+D_n=0$$

and then (i) and (ii) are directly satisfied

Thus, we have

$$\frac{C_n}{\cosh(n-\frac{1}{2})(a+\beta)}=\frac{D_n}{-\cosh(n+\frac{3}{2})(a+\beta)}$$

$$=\frac{A_n+B_n}{\cosh(n-\frac{1}{2})(a+\beta)-\cosh(n+\frac{3}{2})(a+\beta)} \quad \cdots \quad (26)$$

Conditions (*iii*) and (*iv*) give the relations

$$\omega a=\frac{1}{c}\sum_{n=0}^{\infty}\sin\left(n+\tfrac{1}{2}\right)\xi\left[\frac{(n-\frac{1}{2})\,A_n}{\sinh(n-\frac{1}{2})(\alpha+\beta)}\right.$$

$$\left.+\frac{(n+\frac{3}{2})\,B_n}{\sinh\,(n+\frac{3}{2})(\alpha+\beta)}\right] \quad \ldots \quad (27)$$

$$\omega' b=\frac{1}{c}\sum_{n=0}^{\infty}\sin\left(n+\tfrac{1}{2}\right)\xi\left[(n-\tfrac{1}{2})\coth\,(n-\tfrac{1}{2})\,(\alpha+\beta)\,A_n\right.$$

$$+(n+\tfrac{3}{2})\coth\,(n+\tfrac{3}{2})\,(\alpha+\beta)\,B_n-(n-\tfrac{1}{2})\tanh\,(n-\tfrac{1}{2})\,(\alpha+\beta)\,C_n$$

$$\left.-(n+\tfrac{3}{2})\tanh\,(n+\tfrac{3}{2})\,(\alpha+\beta)\,D_n\right] \ldots \quad (28)$$

Having regards to (26) these can be written as

$$\omega ac=\sum_{n=0}^{\infty}(\lambda_n\,A_n+\mu_n\,B_n)\sin\,(n+\tfrac{1}{2})\,\xi, \quad \ldots \quad (29)$$

$$\omega' bc=\sum_{n=0}^{\infty}(\lambda'_n\,A_n+\mu'_n\,B_n)\sin\,(n+\tfrac{1}{2})\,\xi, \quad \ldots \quad (30)$$

where

$$\left.\begin{aligned}
\lambda_n&=\frac{n-\frac{1}{2}}{\sinh\,(n-\frac{1}{2})\,(\alpha+\beta)},\\
\mu_n&=\frac{(n+\frac{3}{2})}{\sinh\,(n+\frac{3}{2})\,(\alpha+\beta)},\\
\lambda'_n&=\frac{-(n-\frac{1}{2})+(n+\frac{1}{2})\cosh 2(\alpha+\beta)-\cosh(2n+1)(\alpha+\beta)}{2\sinh\,(\alpha+\beta)\sinh\,(n+\frac{1}{2})\,(\alpha+\beta)},\\
\mu'_n&=\frac{(n+\frac{3}{2})-(n+\frac{1}{2})\cosh 2\,(\alpha+\beta)-\cosh\,(2n+1)(\alpha+\beta)}{2\sinh\,(\alpha+\beta)\sinh\,(n+\frac{1}{2})\,(\alpha+\beta)}.
\end{aligned}\right\}\ldots \quad (31)$$

We can always expand a constant in a Fourier's series; we have in fact

$$k=\frac{4k}{\pi}\left(\frac{\sin\frac{\xi}{2}}{1}+\frac{\sin\frac{3\xi}{2}}{3}+\frac{\sin\frac{5\xi}{2}}{5}+\ldots\right) \quad \ldots \quad (32)$$

Therefore in order to satisfy (29) and (30), we make

$$\lambda_n \, A_n + \mu_n \, B_n = \frac{4\omega ac}{\pi} \, \frac{1}{2n+1} \qquad \ldots \quad (33)$$

$$\lambda'_n \, A_n + \mu'_n \, B_n = \frac{4\omega' bc}{\pi} \quad \frac{1}{2n+1} \qquad \ldots \quad (34)$$

Therefore

$$A_n = \frac{4c}{(2n+1)\pi} \, \frac{\omega a \mu'_n - \omega' b \mu_n}{\lambda_n \mu'_n - \lambda'_n \mu_n} \qquad \ldots \quad (35)$$

$$B_n = \frac{4c}{(2n+1)\pi} \, \frac{\omega b \lambda_n - \omega a \lambda'_u}{\lambda_n \mu' - \lambda_n \mu_n} \qquad \ldots \quad (36)$$

Obituary Notices

I

THE LATE MR. CHANDRASHEKHAR SIRCAR.

We regret to record the death of Mr. Chandrashekhar Sircar, the leading member of the Bhagalpur Bar and one of the foremost lawyers in Behar and Orissa. Mr. Chandrashekhar Sircar was a distinguished graduate of the Calcutta University having headed the list of successful candidates at the M.A. Examination in Mathematics in the year 1878. He was a self-made man and rose from humble circumstances to affluence by dint of merit of a pre-eminently high order. He was a life-member of the Calcutta Mathematical Society.

II

THE LATE PRINCIPAL RAMENDRASUNDAR TRIVEDI.

By the death of Principal Ramendrasundar Trivedi, Bengal has lost one of her great educationists. Ramendrasundar obtained the Master's Degree of the Calcutta University in Natural and Physical Science in 1887 and occupied the first place in the first class. Next year, Mr. Trivedi greatly distinguished himself by winning the Premchand Roychand Scholarship which is very appropriately called 'the blue ribbon' of the University. His contribution to the Bengali scientific literature has considerably enriched that language. His 'Prákriti' and 'Jignāshā' contain some of the most beautiful essays on science, philosophy and religion. Mr. Trivedi was intimately associated with the Sāhityá Párishád. For many years he was Principal of the Ripon College, a Fellow of the Calcutta University and a member of the Calcutta Mathematical Society. Last year, at the request of the University authorities he delivered a course of lectures on the Vedas which was very much appreciated by the students. He was preparing to follow up the subject this year with another series of lectures on "Vedic Sacrifices." But the cruel hand of death was on him and he passed away.

New Methods in the Geometry of a Plane Arc.

II—Cyclic points and normals.

BY

S. Mukhopadhyaya.

[*Read January 26th, 1919.*]

Introductory.

A *simple arc* for the purposes of the present paper will be defined as follows :

(*i*) It is a continuous curve bounded by two extreme points.

(*ii*) It has a tangent at each point which turns continuosly along the arc in the same direction.

(*iii*) No straight line meets it at more than two points.

(*iv*) The circle determined by any three points of the arc is always finite and varies continuously with the determining points, in other words, the arc possesses continuous curvature.

A *simple oval* may be defined to be a closed curve of which every arc is simple.

An arc NPQ of a circle C intersecting a simple arc S at P will be said to *in*-cross S at P if it crosses from the convex to the concave side at P, and to *ab*-cross S at P if it passes from the concave to the convex side.

A circle C is said to have *ordinary* contact with S at P if it passes through only two consecutive points of S at P. A circle having ordinary contact with S at P will be said to have *in*-contact with S at P if it falls on the concave side of S at P and to have *ab*-contact with S if it falls on the convex side of S at P.

A circle C passing through only three consecutive points at P will be said to have *cru*-contact with S at P.

If NPQ be an arc of a circle having *cru*-contact with S at P, then NPQ will be said to *in*-cross or *ab*-cross S at P according as NPQ passes from convex to concave side or from concave to convex side of S at P. If NPQ in-crosses S at P then we may say that the *portion* NP has *ab*-contact with S at P and the *portion* PQ has *in*-contact with S at P.

If a circle C passes through four consecutive points of S at P then P is called a cyclic point of S and the circle C may be said to have cyclic contact with S at P. A cyclic point will be called *in*-cyclic or *ab*-cyclic* according as the circle C falls on the concave or convex side of S at P.

It may be observed here that according to our New Methods a circle cannot meet a *fixed* curve at more than four consecutive points. This matter will be discussed in a subsequent paper.

We will denote an arc of S between P_1 and P_2 by S_{12} and an arc of a circle C from P_1 to P_2 by C_{12} and so on.

A circular arc C_{12} will be called *cyclic to* S between P_1 and P_2 if it meets S in two or more points between P_1 and P_2. One or two or even three of these extra points may be consecutive to P_1 or P_2 giving rise to an ordinary contact or a cru-contact or a cyclic contact of C_{12} with S at P_1 or P_2.

A circular arc C_{12} which is cyclic to S between P_1 and P_2 will be either *in*-cyclic or *ab*-cyclic or *cru*-cyclic to S between P_1 and P_2. If C_{12} produced *in*-crosses S both at P_1 and P_2 it is *in*-cyclic. If C_{12} produced *ab*-crosses S at both P_1 and P_2 it is *ab*-cyclic. If C_{12} produced *in*-crosses at one and *ab*-crosses at the other of the two points P_1 and P_2 it is *cru*-cyclic.

If C_{12} has ordinary contact or cyclic contact with S at P_1 or P_2 or at both of these points then instead of *in*-crossing or *ab*-crossing at these points we shall read *in*-contact or *ab*-contact in the above definitions.

A fundamental theorem which has been established in my first paper and of which we shall make frequent use in the present paper may now be re-stated in the following form:

If a circular arc C_{12} is in-cyclic to a simple arc S between P_1 and P_2 then there exists at least one in-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is ab-cyclic to S between P_1 and P_2 then there exists at least one ab-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is

*In my first paper (See Bulletin Calcutta Mathematical Society Vol. I. Part I), I have, it is believed for the first time, distinguished the two kinds of cylic points and called them *in*-cyclic and *ex*-cyclic. The latter kind of cyclic points I have preferred to call here *ab*-cyclic, as the prefix *ab*-seems to me more euphonic and significant.

cru-cyclic to S between P_1 and P_2 then there exists at least one in-cyclic and one ab-cyclic point on S between P_1 and P_2.

It may be observed that according to our New Methods there cannot be such a thing as a cru-cylic point on S, that is, a point at which a circle meets S at five consecutive points. It may also be mentioned in connection with the above theorem that when we say a cyclic point exists on S between P_1 and P_2 we mean to exclude P_1 and P_2.

THEOREM I.

P_1, P_2, P_3 are three points taken in order on a simple arc S and the normals at P_1, P_2, P_3 meet at a common point O, which is not the centre of curvature of S at P_2 and which is towards the concave side of S. Then there will be at least one cyclic point X on S between P_1 and P_3, provided none of the angles $P_1 O P_2$ and $P_2 O P_3$ exceed two right angles. The point X will be *in*-cyclic or *ab*-cyclic according as $O P_2$ is a *maximal* or *minimal* normal, that is, according as $O P_2$ is a maximum or minimum radius vector from O to S.

Case I. When each of the angles $P_1 O P_2$ and $P_2 O P_3$ is less than two right angles.

We may suppose without any loss of generality that $O P_1$ and $O P_3$ are the two normals from O to S, nearest to $O P_2$ on either side, for if X lie between the feet of two nearer normals on either side, much more will it lie between the feet of two further normals on either side.

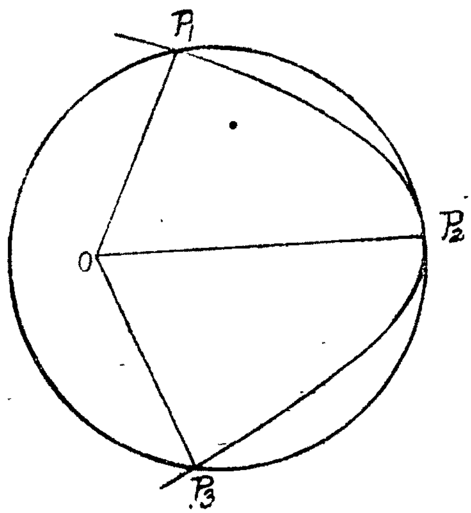
Suppose $O P_2$ is a maximal normal. Then $O P_2$ is the maximum radius vector from O to S in the whole neighbourhood $P_1 P_2 P_3$ and is therefore smaller than both $O P_1$ and $O P_3$. Draw a circle through P_1 to touch S at P_2. We will denote this circle by C and the arc of this circle from P_1 to P_2 by C_{12}. Then since angle $P_1 O P_2$ is less than two right angles and $O P_1$ is less than $O P_2$ the arc C_{12} meets $O P_1$ at an obtuse angle and therefore when produced towards P_1 will *in*-cross S at P_1.

Similarly draw a circle C′ through P_3 to touch S at P_2. Denote the arc of this circle from P_3 to P_2 by C'_{32}. Then C'_{32} will meet $O P_3$ at an obtuse angle and therefore when produced towards P_3 will *in*-cross S at P_3.

Then either C and C′ will coincide or one will fall within the other.

If C and C′ coincide then the circular arc $P_1 P_2 P_3$ will meet S *in*-cyclically between P_1 and P_3 and therefore there must be at least one *in*-cyclic point on S between P_1 and P_3. *See fig. I.*

Fig. I.



If C and C′ do not coincide, then one will fall within the other. Suppose C falls within C′.

The circle C will have either *in*-contact or *ab*-contaet or *cru*-contact with S at P_2.

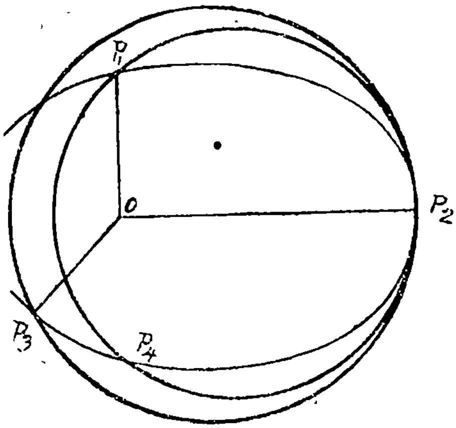
If C has *in*-contact with S at P_2 then C_{12} must cross S_{12} somewhere between P_1 and P_2, and consequently C_{12} will meet S_{12} *in*-cyclically between P_1 and P_2. Thus there is an *in*-cyclic point on S between P_1 and P_2.

If C has *ab*-contact with S at P_2 then C_{12} produced towards P_2 will pass between S_{23} and C'_{23}, *i.e.* C will enter at P_2 the space bounded by S_{23} and C'_{23}. C must therefore come out of this space at some point P_4 (*See fig. II*) on S_{23} between P_2 and P_3. Thus C meets S *in*-cyclically between P_1 and P_4. Consequently there is an *in*-cyclic point on S between P_1 and P_4.

If C has *cru*-contact with S at P_2 then C_{12} will either *in*-cross S at P_2 or *ab*-cross S at P_2. In the former case there will be an *in*-cyclic point on S between P_1 and P_2 and in the latter case an *in*-cyclic point between P_2 and P_4.

Next suppose that OP_2 is a minimal normal. In this case we can prove, by reasoning exactly similar that there is at least one *ab*-cyclic point on S between P_1 and P_3.

Fig. II.

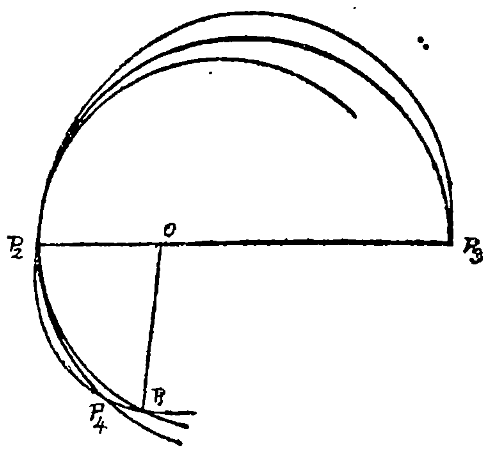


Case II. When angle P_1 O P_2 is less than two right angles and angle P_2 O P_3 is equal to two right angles.

Suppose OP_2 is a minimal normal so that OP_1 and OP_3 are each greater than OP_2.

Draw a circle C to pass through P_1 and to touch S at P_2. Then because the angle P_1 OP_2 is less than two right angles and OP_1 is greater than OP_2 the arc C_{12} will meet OP at an acute angle and consequently C_{12} when produced towards P_1 will *ab*-cross S at P_1. *See fig. III.*

Fig. III.



The circle C will either have *ab*-contact or *in*-contact or *cru*-contact with S at P_2. If C have *ab*-contact with S at P_2 then C will cross S between P_1 and P_2 and consequently there will be an *ab*-cyclic point on S between P_1 and P_2. If C have *cru*-contact with S at P_2 then C_{12} will either *ab*-cross S at P_2 or *in*-cross S at P_2. In the latter case C_{12} must cross S between P_1 and P_2. So that in either case there will be an *ab*-cyclic point on S between P_1 and P_2.

If C have *in*-contact with S at P_2 then C will either meet S_{23} between P_2 and P_3 at some point P_4 or fall below S_{23}. In the former case there is an *ab*-cyclic point on S between P_1 and P_4.

In the latter case draw the circle C' or rather the semi-circular arc C'_{32} to touch S at P_2 and P_3. If C'_{32} have *ab*-contact with S at P_2 and P_3 then an *ab*-cyclic point on S between P_2 and P_3 is assured. If C'_{32} have contacts *ab* and *in* or *in* and *ab* at P_2 and P_3 then C'_{32} must necessarily cross S between P_2 and P_3 and an *ab*-cyclic point on S between P_2 and P_3 is assured.

If C'_{32} have *in*-contact with S at P_2 and P_3 then C' will enter the space formed by S_{12} and C_{12} at P_2 and consequently *ab*-cross S at some point P_4 between P_1 and P_2. *See fig III.* Consequently there will be an *ab*-cyclic point on S between P_3 and P_4.

Thus on the supposition that OP_2 is a minimal normal there is always an *ab*-cyclic point on S between P_1 and P_3.

If we had supposed OP_2 to be a maximal normal we could prove by similar reasoning that there is always an *in*-cyclic point on S between P_1 and P_3.

COROLLARY TO THEOREM I.

If the normals to S at P_1 and P_3 meet at P_2 then there is at least one ab-cyclic point on S between P_1 and P_3. If the normals at P_1 and P_2 meet at P_3 then there is an ab-cyclic or in-cyclic point on S between P_1 and P_3 according as P_2P_3 is a minimal or a maximal normal.

This corollary follows from theorem I if we make one of the three normals $OP_1 OP_2 OP_3$ vanish. It can however be proved directly quite easily.

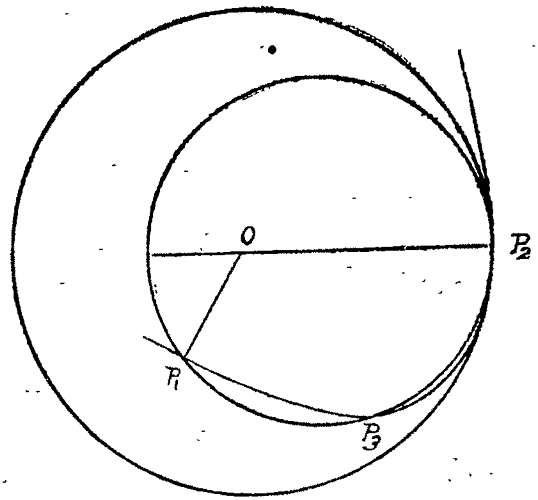
THEOREM II.

If OP_1 and OP_2 be two successive normals to a simple arc S from a point O, on the concave side of S, including between them an angle not

exceeding two right angles, and if O be the centre of curvature of S at P_2, then there is at least one cyclic point on S between P_1 and P_2, which is in or- ab- according as OP_1 is less or greater than OP_2.

Suppose angle $P_1\ OP_2$ is less than two right angles and OP_1 is less than OP_2. *See fig. IV.*

Fig. IV.



Draw a circle C to pass through P_1 and touch S at P_2. Then the arc C_{21} of this circle will meet OP_1 at an obtuse angle and consequently *in*-cross S at P_1.

Draw a circle C′ with centre O and radius OP_2. Then C′ is the circle of curvature of S at P_2 and touches C externally at P_2. The circular arc C_{21} will therefore have *in*-contact with S at P_2. Consequently C_{21} must cross S at some point P_3 between P_1 and P_2. Thus C_{21} is *in*-cyclic to S between P_1 and P_2 which ensures the existence of an *in*-cyclic point on S between P_1 and P_2.

If we suppose the angle $P_1\ OP_2$ to be equal to two right angles, then C_{12} will have in contact with S at P_2 and either *in*- or *ab*-contact with S at P_1. In the former case C_{12} is in-cyclic to S between P_1 and P_2 and in the latter case C_{12} is cru-cyclic to S between P_1 and P_2. In either case the existence of an in-cyclic point on S between P_1 and P_2 is assured.

If OP_1 be greater than OP_2 the existence of an *ab*-cyclic point on S between P_1 and P_2 can be similarly established.

In this theorem we have supposed O to be the centre of curvature of S at P_2. The centre of curvature of S at P_1 will in general not be at O but it can be also at O as a special case.

COROLLARY TO THEOREM II.

If the centre of curvature of S at a point P_1 *be a point* P_2 *which is on S then there is at least one in-cyclic point on S between* P_1 *and* P_2.

The three following theorems follow at once from theorems I and II and their corollaries.

THEOREM III.

If from a point O on the concave side of a simple arc S it is possible to draw n normals to S and if the angle between any pair of successive normals do not exceed two right angles then there are at least n—2 cyclic points on S between the feet of the first and the last normal.

THEOREM IV.

If from a point O interior to a simple oval it is posible to draw n normals to the oval and if the angle between any pair of successive normals do not exceed two right angles then there are at least n cyclic points on the oval.

THEOREM V.

If from a point O on a simple oval it be possible to draw n normals to S, excluding the normal at O, then there are at least n + 1 cyclic points on the oval.

In the above theorems if O be the centre of curvature at O for any normal OP then such a normal has to be counted twice. If in addition the point P be a cyclic point then the normal OP has to be counted thrice.

Origin of the Indian Cyclic Method for the Solution of $Nx^2+1=y^2$.

BY

P. C. SEN-GUPTA.

[*Read October 6th, 1918.*]

1. The object of the present paper is to discuss the probable origin of the "Cyclic Method" (*Chakrabala*) for the solution of $Nx^2+1=y^2$ in rational integers as given in Bhaskara's *Vijaganita*. Two hypotheses have been advanced as regards its origin: * first that the method has an ultimate Greek source and secondly that it is purely Indian. I shall first discuss the former view and shall next show that it is untenable in the light of the reasons which, I trust, are put forth herein for the first time.

2. The Cyclic Method and other Rules.

(*a*) The Cyclic Method as given by Bhaskara may be stated as follows :—

† To solve $Nx^2+1=y^2$, where N is a non-square integer; start with a relation of the form:

$Na^2+k=b^2$, where a, b, k are all simple integers; derive from it the following relation:

$$N\left(\frac{a\alpha+b}{k}\right)^2+\frac{\alpha^2-N}{k}=\left(\frac{b\alpha+Na}{k}\right)^2,$$

where α and $\dfrac{a\alpha+b}{k}$ are integers and α^2-N has the least value.

Repeat the operation with the new relation and obtain a fresh relation in the same form in the same way: continue to proceed in the

* Sir T. L. Heath's Diophantus (1910), page 281, also the references mentioned therein, viz, Tannery "Sur la mesure du cercle d'Archimède" in Mém de la soc. des sciences phys et nat. de Bordeaux, IIe Sér. IV, 1882. page 325; *cf.* Konen, pp. 27-28; Bibliotheca Mathematica, VI 3, 1905-6, pp. 271-73.

† Colebrooke's Indian Algebra (1817), page 175. Heath's Diophantus page 283.

same manner till *integral roots* are obtained with any of the numbers 4, 2 or 1 for the additive. Now apply "composition" [*i.e.*, the rule (*b*) given below] for the solution of $Nx^2+1=y^2$.

(*b*). If $Na^2+k=b^2$ and $Na'^2+k'=b'^2$, then will

$$N\,(ab'\pm a'b)^2+kk'=(Naa'\pm bb')^2.$$

‡ This lemma was first given by Brahmagupta in A.D. 628.

(*c*). $x=\dfrac{2\gamma}{N-\gamma^2}$, $y=\dfrac{N+\gamma^2}{N-\gamma^2}$ is a solution of $Nx^2+1=y^2$.

Neither Brahmagupta nor Bhāskara has given any formal method of deduction of the above rules. But this need not create any surprise or surmise of a Greek Origin. The first discoverers often get their results by intuition and trial.

3. Illustration of the Cyclic Method.

In the cyclic rule stated above, when a and $\dfrac{aa+b}{k}$ are both integers, it is not difficult to prove, (1) that a, k and b may be taken to be prime to one another and (2) that $\dfrac{a^2-N}{k}$ and $\dfrac{ba+Na}{k}$ are both integers. The rules (*b*) and (*c*) are also readily proved. A numerical example is given below to illustrate the cyclic rule.

To solve $67\,x^2+1=y^2$.

Here the relation to start with is

$$67\times1^2-3=8^2. \qquad \ldots \qquad (1)$$

We are to solve $\dfrac{a+8}{-3}=\beta$ in integers. the suitable solution which makes a^2-N the least, is $a=7$ and $\beta=-5$, whence $\dfrac{a^2-N}{k}=6$, $\dfrac{ba+Na}{k}=-41$, *i.e.*, the relation arrived at is

$$67\times5^2+6=41^2 \qquad \ldots \qquad (2).$$

We are now to solve $\dfrac{5a+41}{6}=\beta$ in integers: the suitable solution

‡ Colebrooke's Indian Algebra (1817), page 363 also Brahma Sphuta Siddhanta, chapter XVIII, 64 and 65; Colebrooke's Indian Algebra, pages 171-172.

is $\alpha=5$ and $\beta=11$, and the new relation becomes

$$67\times 11^2-7=90^2. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Similarly the next relation deduced from (3) is

$$67\times 27^2-2=221^2. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

As the additive in (4) has become -2, the cyclic method need not be followed any more; the equation is very expeditiously solved by applying the rule of Brahmagupta [§ 2 (b)] thus :—

We have $67\times 27^2-2=221^2$

and $67\times 27^2-2=221^2$.

$\therefore \quad 67(2\times 27\times 221)^2+4=(67\times 27^2+221^2)^2$

or $67\times 5967^2+1 \quad =48842^2, \quad \ldots \quad \ldots \quad \ldots \quad (5)$

hence $x=5967$ and $y=48842$ is a solution of $67x^2+1=y^2$. From the last numerical relation repeated, application of Brahmagupta's rule leads to any number of solutions.

4. Hypothesis of ultimate Greek Origin.

(a) The "Diophantine Method" and the Indian Method.

M. Tannery has held that probably somewhere in one of the lost books of the Arithmetica, Diophantus solved the equation $x^2-Ay^2=1$. He has shown how from the Diophantine method, if one solution (p, q) of $x^2-Ay^2=1$ is known a more general solution may be found :—

"§ Put $p_1=mx-p$, $q_1=x+q$,

and suppose

$$p_1{}^2-Aq_1{}^2=m^2x^2-2mxp+p^2-Ax^2-2Aqx-Aq^2=1,$$

therefore (since $p^2-Aq^2=1$) $x=\dfrac{2mp+Aq}{m^2-A}$, and by substitution in the expression for $p_1 q_1$, we have

$$p_1=\frac{(m^2+A)p+2Amq}{m^2-A},\ q_1=\frac{2mp+(m^2+A)q}{m^2-A}, \text{ and in fact}$$

$p_1{}^2-Aq_1{}^2=1$.

If an integral solution is wanted, one way of obtaining it is to substitute u/v for m where $u^2-Av^2=1$, *i.e.*, where u, v is another

§ Heath's Diophantus, page 280.

solution of the original equation, and we then have,

$p_1=(u^2+Av^2)p+2Auvq,\ q_1=2puv+(u^2+Av^2)q.$

"But this is all that we can get out of Dophantus as we have him, and it will be observed that here too *we must have ascertained two solutions of the one equation, or one solution of it and a solution of an auxiliary equation before we can apply the method.*"

It is evident from the above that there is hardly any thing common between this "Diophantine Method" and the Indian Cyclic method. It is so very imperfect that it cannot proceed without *two solutions* of the same equation. In the Indian method when *one solution* of $x^2-Ay^2=1$ is known, any number of solutions may be found by Brahmagupta's rule and that the Indian method in this case is exactly the same as the modern method. Clearly then the "ultimate Greek origin" does not lie in Diophantus's Arithmetica.

(*b*) The Archimedian approximations to a surd and the cyclic method.

Again M. Tannery's method of showing how|| "from the Greek manner of deducing from approximation to surd a nearer approximation, it is possible by simple steps to pass to the Indian method", need not be taken as a reason for considering the indebtedness of the Indian cyclic method to any ultimate Greek origin, as there is nothing on record in the works of Archimedes which shows that he actually discovered it or applied it to the solution of $Nx^2+1=y^2$. Although I have to admit that I have not yet had access to the reference given in Dr. Heath's work, I have been able to discover a way of deriving the cyclic rule from the Archimedian method of approximating to the value of a surd, which, I trust, will not differ much from M. Tannery's. Both the Archimedian method of finding approximations to a surd and the deduction of the cyclic rule from it are exhibited below.

(1) Archimedian Method of finding $\sqrt{N}$ approximately.

§ Hültsch proves that Archimedes "discovered and proved" that

$$a\pm\frac{b}{2a}>\sqrt{a^2\pm b}>a\pm\frac{b}{2a\pm1}.$$

|| Heath's Diophantus, page 281.

§ Works of Archimedes (Heath, 1897), page lxxxii et sec.

Hence according to Archimedes

$\sqrt{a^2 \pm b}$ is approximately

$$= a \pm \frac{b}{2a} \text{ or } a \pm \frac{b}{2a \pm 1}.$$

As an illustration let us find the approximate values of $\sqrt{67}$.

$$\text{Here } \sqrt{67} = \sqrt{8^2+3} = 8 + \frac{3}{2 \times 8}, \left(\because \sqrt{a^2 \pm b} = a \pm \frac{b}{2a}\right)$$

$$= 8 + \frac{1}{5} \text{ nearly,} \quad \ldots \quad \ldots \quad (1)$$

and $\left(8\frac{1}{5}\right)^2 - 67 = \frac{6}{25}$, whence $y=41$ and $x=5$ is a solution of

$$y^2 - 67x^2 = 6.$$

Hence

$$\sqrt{67} = \sqrt{\left(\frac{41}{5}\right)^2 - \frac{6}{25}} = \frac{41}{5} - \frac{6/25}{\frac{2 \times 41}{5} - 1}, \left(\therefore \sqrt{a^2 \pm b} = a \pm \frac{b}{2a \pm 1}\right)$$

$$= \frac{41}{5} - \frac{6}{385} = \frac{90}{11} \text{ nearly;} \quad \ldots \quad \ldots \quad (2)$$

and $\left(8 + \frac{2}{11}\right)^2 - 67 = -\frac{7}{121}$, whence $y=90$, $x=11$ is a solution of $y^2 - 67x^2 = -7$.

$$\text{Again } \sqrt{67} = \sqrt{\left(\frac{90}{11}\right)^2 + \frac{7}{121}} = \frac{90}{11} + \frac{7/21}{\frac{2 \times 90}{11} + 1} \text{ nearly}$$

$$= \frac{90}{11} + \frac{1}{11 \times 27} \text{ nearly}$$

$$= \frac{221}{27} = 8 + \frac{5}{27} \ldots \quad \ldots \quad (3)$$

From the last approximation it is seen that $\left(8 + \frac{5}{27}\right)^2 - 67 = \frac{2}{27^2}$ or $y=221$, $x=27$ is a solution of $y^2 - 67x^2 = -2$.

Thus with some difficulty we arrive at the four approximate values of $\sqrt{67}$ by the Archimedian process, *viz.*, 8, $\frac{41}{5}$, $\frac{90}{11}$ and $\frac{221}{27}$, and in passfrom one approximation to another we solve an equation of the form $y^2 - Nx^2 = k$. If however we develop $\sqrt{67}$ as a continued fraction,

the first five convergents are $\frac{3}{1}, \frac{41}{5}, \frac{90}{11}, \frac{131}{16}$ and $\frac{222}{27}$, four of which are obtained from Archimedes's rules. It is probable that he did really solve some numerical equations by his methods.

(2) Deduction of the Cyclic Method from the Archimedian process of finding $\sqrt{N}$ approximately.

From what has been shown above the Archimedian approximation proceeds either by

$$\sqrt{a^2 \pm b} = a \pm \frac{b}{2a} \text{ approximately,}$$

$$\text{or } = a \pm \frac{b}{2a \pm 1} \text{ approximately.}$$

Now when we have a relation, $Na^2 + k = b^2$, $\frac{b}{a}$ is evidently the first approximation to $\sqrt{N}$. In proceding to the next approximation we have

$$N = \frac{b^2}{a^2} - \frac{k}{a^2}.$$

$$\therefore \sqrt{N} = \sqrt{\frac{b^2}{a^2} - \frac{k}{a^2}} = \frac{b}{a} - \frac{\frac{k}{a^2}}{\frac{2b}{a} - 1} \text{ approximately,}$$

$$= \frac{2b^2 - ab - k}{a(2b - a)}.$$

Now $(2b^2 - ab - k)^2 - Na^2(2b - a)^2 = k(k - 2ab + b^2)$
$= k\{(b - a)^2 - Na^2\}$.

Let $b - a = a\alpha + \gamma$,

$\therefore \; k\{(b-a)^2 - Na^2\} = k(a^2\alpha^2 - Na^2)$ rejecting γ,
$= ka^2(\alpha^2 - N)$.

Again $a(2b - a) = a\{b + (b - a)\} = a(b + a\alpha)$ rejecting γ as before.
and $2b^2 - ab - k = b(b - a) + Na^2 = a(b\alpha + Na)$ rejecting γ as before.

Hence we should have

$a^2(b\alpha + Na)^2 - Na^2(b + a\alpha)^2 = ka^2(\alpha^2 - N)$, which is easily seen to be correct by actual work. Now divide both sides by k^2a^2, and we get

$$\left(\frac{b\alpha + Na}{k}\right)^2 - N\left(\frac{a\alpha + b}{k}\right)^2 = \frac{\alpha^2 - N}{k}, \text{ which is exactly the}$$

cyclic rule.

This is indeed one way of arriving at the Indian rule; but the steps of putting $b-a=a\alpha+\gamma$ and rejecting γ although simple enough, are not natural. If M. Tannery has indeed deduced the cyclic rule by the above process or by a process very like the above and has imagined an "ultimate Greek" origin to the Indian cyclic rule, I feel inclined to think that he has not done sufficient justice to the Indian Algebraists. Neither Brahmagupta nor Bhaskara does anywhere connect the finding of the approximate values of $\sqrt{N}$ with the solution of $Nx^2+1=y^2$. A simpler method of arriving at the rule is given below, but this also does not seem to be the natural way of its discovery.

Let $Na^2+k=b^2$ and $Na'^2+k'=b'^2$ where a' and b' are respectively greater then a and b; let us suppose that

$$a'=a\alpha+\gamma_1,$$

$$b'=b\alpha+\gamma_2 \text{ when } \alpha, \gamma_1, \text{ and } \gamma_2 \text{ are undetermined.}$$

$$\text{Here } b'^2-Na'^2=(b\alpha+\gamma_2)^2-N(a\alpha+\gamma_1)^2$$
$$=\alpha^2(b^2-Na^2)+2\alpha(b\gamma_2-Na\gamma_1)+(\gamma_2{}^2-N\gamma_1{}^2)$$

In order to simplify the right hand expression, assume $\gamma=b$ and $\gamma_2=Na$, so that the middle term disappears and we get

$$(b\alpha+Na)^2-N(a\alpha+b)^2=k(\alpha^2-N),$$

$$\therefore \left(\frac{b\alpha+Na}{k}\right)^2-N\left(\frac{a\alpha+b}{k}\right)^2=\frac{\alpha^2-N}{k}, \text{ which is the cyclic rule.}$$

Similarly other methods of arriving at the Indian rule may not be impossible and still not be in any way connected with its origin. We now turn to the other theory.

5. Hypothesis of a purely Indian Origin.

A most unanswerable argument for the Indianness of the method lies in its new-born naturalness and simplicity. Hankel who strongly advocates the Indian Origin of the method surmises that the Indians probably deduced it as follows :—

(a) Hankel's Method.

* Let $Na^2+k=b^2$ and $Na'^2+k'=b'^2$, then by Brahmagupta's rule we have

$$N(ab'-a'b)^2+kk'=(Naa'-bb')^2$$

Put $ab'-a'b=1$ and $Naa'-bb'=\alpha$

$$\therefore N+kk'=\alpha^2, \text{ or } k'=\frac{\alpha^2-N}{k}.$$

* Heath's Diophantus, page 284.

Next determine a' and b' from the equations

$$ab'-a'b-1=0 \text{ and } Naa'-bb'-a=0,$$

$$\text{and } a'=\frac{aa+b}{-k} \text{ and } b'=\frac{ba+Na}{-k},$$

$$\text{or } N\left(\frac{aa+b}{k}\right)^2+\frac{(a^2-N)}{k}=\left(\frac{ba+Na}{k}\right)^2.$$

(b) Method given in M.M. Sudhakara Dvivedi's edition of Bhaskara's *Vijaganita.*

$$\text{Let } Na^2+k=b^2, \qquad \dots \quad (1)$$

and we have identically

$$N\times 1^2+(a^2-N)=a^2, \qquad \dots \quad (2)$$

∴ by Brahmagupta's lemma we get

$$N(aa+b)^2+k(a^2-N)=(Na+ba)^2,$$

$$\therefore\ N\left(\frac{aa+b}{k}\right)^2+\frac{a^2-N}{k}=\left(\frac{Na+ba}{k}\right)^2,$$ which is the *Chakra bala* or the Indian cyclic rule.

6. The above (b) method, as far as I have been able to ascertain seems to have been followed by all the pupils of the late M.M. Bapudev Sastri and the late Pandit Sudhakara Dvivedi. I feel inclined to believe that this elegant method is the true Indian method as transmitted through generations of *gurus.* The third rule given in §2 is easily deduced from it.

We have as before

$$N\times 1^2+(a^2-N)=a^2$$

$$\text{and } N\times 1^2+(a^2-N)=a^2,$$

∴ by the lemma of Brahmagupta, we get

$$N(2a)^2+(a^2-N)^2=(N+a^2)^2,$$

$$\text{or } N\left(\frac{2a}{a^2-N}\right)^2+1=\left(\frac{N+a^2}{a^2-N}\right)^2, \text{ whence } x=\frac{2a}{a^2-N}$$

and $y=\left(\frac{N+a^2}{a^2-N}\right)$ is a solution of $Nx^2+1=y^2$.

It is thus seen that to arrive at the Indian cyclic rule it is not at all necessary to determine an approximation to $\sqrt{N}$ either by the Archimdian method or by any other method. It is further evident that the rules are immediate deductions from the lemma of Brahmagupta, and the sole credit of finding a method for the solution of $Nx^2+1=y^2$ belongs to him.

On the motion of an ellipsoid of revolution in a viscous fluid in the light of Prof. Oseen's objection to Stokes's treatment of the case of the sphere.

BY

BHOLANATH PAL.

[*Read March 3rd, 1918.*]

INTRODUCTION.

The motion of a sphere in a viscous fluid has been investigated by various writers including Stokes,[1] Profs. Whitehead,[2] Oseen,[3] Lamb,[4] and Burgess,[5] the results obtained being more or less satisfactory according to the degree of approximation to which the differential equations are satisfied.

In the pres nt paper, I propose (1) to obtain the solution of the problem of the motion of translation of an ellipsoid of revolution of small ellipticity in a viscous fluid, the method adopted being similar to that of Prof. Lamb for treating the corresponding problem in the case of the sphere, and (2) to show how the results obtained by me although different in some respects from those given by Oberbeck,[6] the only important writer who investigated the ellipsoidal problem before me, are free from any objection similar to that pointed out by Prof. Oseen in Stokes's treatment of the spherical problem.

In Art. 1, I reproduce the objection raised by Prof. Oseen to Stokes's solution of the spherical problem, in Arts. 2 and 3, I

[1] See his "*Scientific Papers*," Vol. 3, p. 1, or *Camb. Transactions*, Vol. 9, p. 8 (1851.)

[2] Whitehead, *Quarterly Journal of Mathematics*, Vol. 23 (1888), pp. 143-152.

[3] Oseen, *Arkiv för Mat. Astr. Och. Fysik*, Bd. 6 (1911), No. 29.

[4] Lamb, *Phil. Mag.*, series 6, Vol. 21 (1911), pp. 112-121.

[5] Burgess, *American Journ. of Math.*, Vol. 38 (1916), pp. 81-96.

[6] Oberbeck, "Ueber stationäre Flüssigkeitsbewegungen mit Berücksichtigung der inneren Reibung," *Crelle's Journal*, Bd 81 (1876), pp. 62-80.

investigate the motion of the ellipsoid of revolution, in Art. 4, I compare my results with those of Oberbeck, in Art 5, I find out the resistance experienced by the ellipsoid.

I should like to express my indebtedness to Dr. Ganes Prasad at whose suggestion I took up, and under whom I carried on the investigation.

Professor Oseen's objection to Stokes's treatment for the case of a sphere.

1. The formula of Stokes for the resistance which a sphere experiences when it moves with constant and infinitely small velocity in a viscous incompressible fluid was proved by its author in the following manner.

The differential equations of Navier for the motion of the fluid, referred to a system of co-ordinates which has its origin in the centre of the sphere and which moves with that of the sphere, are

$$\left.\begin{aligned}
\rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right\}&=-\frac{\partial p}{\partial x}+\mu\nabla^2 u,\\
\rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right\}&=-\frac{\partial p}{\partial y}+\mu\nabla^2 v,\\
\rho\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\}&=-\frac{\partial p}{\partial z}+\mu\nabla^2 w,\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0.
\end{aligned}\right\} \qquad (1)$$

The corresponding auxiliary conditions are, for $R=\sqrt{x^2+y^2+z^2}=\infty$, $u=-U$, $v=0$, $w=0$; for $R=a$, $u=0$, $v=0$, $w=0$, if U be the velocity of the sphere and a its radius and if the x-axis be identical with the direction of the motion with the sphere.

We suppose that the motion of the fluid brought about by the sphere is stationary. Thus the first members in the three first equations fall away. Further it is clear that if at all the three functions u, v, w exist which satisfy the differential equations and the auxiliary conditions and which everywhere outside the sphere are, together with their derivatives of the first two orders, finite and continuous functions

of x, y, z, then $u, v, w, \frac{\partial u}{\partial x}, \ldots\ldots \frac{\partial w}{\partial z}$ must diminish everywhere to zero as U tends to zero. It is therefore reasonable to suppose that if U is small the so-called quadratic members $u\frac{\partial u}{\partial x}$ .... must be of higher order of smallness than the members $\frac{\partial p}{\partial x}$, ... $\nabla^2 u$, ... and that one may consequently neglect the quadratic members. If one does this the differential equations receive the comparatively small forms

$$\mu\nabla^2 u=\frac{\partial p}{\partial x},$$

$$\mu\nabla^2 v=\frac{\partial p}{\partial y},$$

$$\mu\nabla^2 w=\frac{\partial p}{\partial z},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=o.$$

This system of differential equations with the auxiliary conditions. for $R=\infty$, $u=-U$, $v=0$, $w=0$; for $R=a$, $u=0$, $v=0$, $w=0$, is very easy to solve. One finds that the functions

$$\left.\begin{aligned}
u&=\frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)x^2-U\left(1-\frac{3}{4}\frac{a}{R}-\frac{1}{4}\frac{a^3}{R^3}\right),\\
v&=\frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)xy,\\
w&=\frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)xz,\\
p&=\frac{3}{2}\mu\frac{aU}{R^3}x
\end{aligned}\right\}\quad(2)$$

satisfy the differential equations as well as the auxiliary conditions. From these formulae one deduces easily Stokes's expression for the resistance of the sphere.

From these one finds that the value of $u\frac{\partial u}{\partial x}$ is

$$-\frac{3}{4}\frac{aU^2x}{R^3}\left(1-\frac{3x^2}{R^2}\right)+ \quad \text{etc.}$$

and

$$\mu\nabla^2 u=\frac{\partial p}{\partial x}=\frac{3}{2}\mu\frac{aU}{R^3}\left(1-\frac{3x^2}{R^2}\right).$$

The ratio of the former to the latter contains x as a factor and therefore its value becomes infinite at a great distance from the sphere. For this reason the formulae in question cannot be regarded as valid at points distant from the sphere.

The motion of the ellipsoid of revolution by Lamb's method.

2. Let the equation of the spheroid be taked in the form $r=a'\{1+\epsilon P_2(\cos\theta)\}$, where ϵ is very small, so that its square and higher powers may be neglected.

Let $\bar{u}$, v, w be the three velocity-components and suppose $\bar{u}=u+U$, so that

$$u=0,\quad v=0,\ w=0 \qquad \text{at infinity,}$$

and

$$u=-U,\ v=0,\ w=0 \qquad \text{on the surface}$$

The hydrodynamical equations accordingly take the forms

$$\left.\begin{aligned} U\frac{\partial u}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,\\ U\frac{\partial v}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v,\\ U\frac{\partial w}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w,\end{aligned}\right\} \quad \dots (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \dots (2)$$

The inertia terms are to some extent taken into account.

The motion is supposed to be steady and the presence of any extraneous force is neglected; thus from the dynamical equations we can easily obtain

$$\nabla^2 p=0, \qquad \dots \quad (3)$$

where p is the hydrodynamical pressure

Thus we can take

$$p=\rho U \frac{\partial \phi}{\partial x} \qquad \dots \quad (3)$$

where ϕ satisfies

$$\nabla^2 \phi=0. \qquad \dots \quad (4)$$

Let us take

$$\left.\begin{aligned} u&=-\frac{\partial \phi}{\partial x}+u'. \\ v&=-\frac{\partial \phi}{\partial y}+v', \\ w&=-\frac{\partial \phi}{\partial z}+w'. \end{aligned}\right\} \qquad \dots \quad (6)$$

Then from the differential equation

$$U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,$$

we have

$$\left(\nabla^2-\frac{U}{\nu}\frac{\partial}{\partial x}\right)u'=0.$$

Putting $k=\dfrac{U}{2\nu}$, this assumes the from

$$\left.\begin{aligned} &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)u'=0. \\ \text{Similarly}\quad &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)v'=0, \\ &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)w'=0, \\ \text{and}\quad &\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0. \end{aligned}\right\} \qquad \dots \quad (7)$$

The spheroid is taken to be ovary, so that its sections perpendicular to the x—axis will be circles and therefore the vortex lines will be circles having the axis of x as a common axis. We may assume

$$\xi=0, \quad \eta=-\frac{\partial X}{\partial z}, \quad \zeta=\frac{\partial X}{\partial y}, \qquad \dots \quad (8)$$

where X is a function of x and ρ (the distance from the axis of x) only and ξ, η, ζ are the components of vorticity.

Then we must have

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0, \qquad \dots \quad (9)$$

Thus

$$\left.\begin{aligned}
2k\frac{\partial u'}{\partial x} &= \nabla^2 u'=\frac{\partial \eta}{\partial z}-\frac{\partial \zeta}{\partial y}=-\left(\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial z^2}\right)\\
&\qquad\qquad\qquad\qquad\quad =\frac{\partial^2 X}{\partial x^2}-2k\frac{\partial X}{\partial x},\\
2k\frac{\partial v'}{\partial x} &= \nabla^2 v'=\frac{\partial \zeta}{\partial x}-\frac{\partial \xi}{\partial z}=\frac{\partial^2 X}{\partial x\,\partial y},\\
2k\frac{\partial w'}{\partial x} &= \nabla^2 w'=\frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial x}=\frac{\partial^2 X}{\partial x\,\partial z}.
\end{aligned}\right\} \qquad \dots \quad (10)$$

Therefore

$$\left.\begin{aligned}
u'&=\frac{1}{2k}\frac{\partial X}{\partial x}-X,\\
v'&=\frac{1}{2k}\frac{\partial X}{\partial y},\\
w'&=\frac{1}{2k}\frac{\partial X}{\partial z}.
\end{aligned}\right\} \qquad \dots \quad (11)$$

In (9), putting $X=e^{kx}X'$, we get

$$(\nabla^2-k^2)X'=0.$$

Therefore equation (9) can be written in the from

$$(\nabla^2-k^2)e^{-kx}X=0. \qquad \dots \quad (12)$$

The solution of this differential equation is

$$e^{-kx}X=C\,\frac{e^{-kr}}{r}, \qquad \dots \quad (13)$$[1]

where C is a constant

From these we have finally

$$\left.\begin{aligned} u&=-\frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial X}{\partial x}-X, \\ v&=-\frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial X}{\partial y}, \\ w&=-\frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial X}{\partial z}. \end{aligned}\right\} \qquad \dots \quad (14)$$

where

$$X=C\,\frac{e^{-k(r-x)}}{r}. \qquad \dots \quad (15)$$

We have $u=0$, $v=0$, $w=0$ at infinity. Therefore ϕ must obvieusly involve only zonal harmonics of negative degrees and we write

$$\phi=\frac{A_o}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)$$

$$+A_3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)+ \qquad \dots \quad (16)$$

where A_o, A_1, A_2, A_3, ... are different constants.

If we take kr very small, we have

$$X=C\left\{\frac{1}{r}-k+\frac{kx}{r}+\frac{k^2x^2}{2\,!}\frac{1}{r}+\ \dots\right\} \qquad \dots \quad (17)$$

[1] See Prof. Lamb's *Hydrodynamics*, § 289, (Third edition).

Thus

$$\left.\begin{aligned}
\frac{1}{2k}\frac{\partial X}{\partial x}-X &= -\frac{C}{2k}\left\{\frac{4}{3}\frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right. \\
&\qquad \left. +\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ldots\right\}, \\
\frac{1}{2k}\frac{\partial X}{\partial y} &= -\frac{C}{2k}\left\{\quad -\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right. \\
&\qquad \left. +\frac{1}{3}kr^2\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)+\ldots\right\}, \\
\frac{1}{2k}\frac{\partial X}{\partial z} &= -\frac{C}{2k}\left\{\quad -\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right. \\
&\qquad \left. +\frac{1}{3}kr^2\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right)+\ldots\right\},
\end{aligned}\right\}\ldots \quad (18)$$

Therefore

$$u=-\frac{\partial}{\partial x}\left[\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ldots\right]$$

$$-\frac{C}{2k}\left[\frac{4}{3}\frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ldots\right], \quad \ldots \quad (19)$$

and similar expressions for v and w.

3. The constants $C, A_0, A_1, A_2, A_3, \ldots\ldots$ can be determined from boundary condition *i.e.*, $u=-U$, for $r=a'[1+\epsilon P_2(\cos\theta)]$.

Therefore making use of the formula

$$P_n(\cos\theta)=(-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right),$$

and retaining only the first power of ϵ we have

$$-U=\frac{A_0}{a'^2}P_1(1-2\epsilon P_2)-\frac{A_1}{a'^3}2!\,P_2(1-3\epsilon P_2)$$

$$+\frac{A_2}{a'^4}3!\,P_3(1-4\epsilon P_2)-\frac{A_3}{a'^5}4!\,P_4(1-5\epsilon P_2)$$

$$+\frac{A_4}{a'^6}\,5!P_5(1-6\epsilon P_2)-\frac{A_5}{a'^7}\,6!P_6(1-7\epsilon P_2)$$

$$+\frac{A_6}{a'^8}\,7!P_7\,(-8\epsilon P_2)-\frac{A_7}{a'^9}\,-8!P_8(1-9\epsilon P_2)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\frac{2C}{3a'}\left(1-\epsilon P_2\right)-\frac{C}{2ka'^2}\,P_1(1-2\epsilon P_2)$$

$$-\frac{C}{3a'}\,P_2(1-\epsilon P_2)-\ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (20)$$

In the above equation using the formula

$$P_2P_n=B_{n+2}\,P_{n+2}+B_n\,P_n+B_{n+2}\,P_{n+2}$$

where

$$B_{n+2}=\frac{3}{2}\,\frac{(n+1)(n+2)}{(2n+1)(2n+3)},$$

$$B_n=\frac{n(n+1)}{(2n+1)(2n+3)},$$

$$B_{n-2}=\frac{3}{2}\,\frac{n(n-1)}{(2n+1)(2n-1)},$$

and comparing the co-efficients of P_0, P_1, P_2,...the following set of equations are obtained to determine the constants :—

$$(i)\quad U+\frac{6\epsilon}{5a'^3}A_1-C\left(\frac{10-\epsilon}{15a'}\right)=0,$$

$$(ii)\quad \left(\frac{5-4\epsilon}{5a'^2}\right)A_0-\frac{4!\cdot 6\epsilon}{35a'^4}A_2-C\left(\frac{5-4\epsilon}{10ka'^2}\right)=0,$$

$$(iii)\quad \frac{2}{7a'^3}(7-6\epsilon)A_1-\frac{5!\cdot 2\epsilon}{7a'^5}A_3+C\left(\frac{7-16\epsilon}{21a'}\right)=0,$$

$$(iv)\quad \frac{6\epsilon}{5a'^2}A_0-\frac{2}{a'^4}\left(3-\frac{16\epsilon}{5}\right)A_2+\frac{6!\cdot 10\epsilon}{33a'^6}A_4-\frac{3\epsilon}{5ka'^2}\,C=0,$$

$$(v)\quad \frac{108\epsilon}{35a'^3}A_1-\frac{24}{a'^5}\left(1-\frac{100}{77}\epsilon\right)A_3+\frac{7!\,45\epsilon}{143a'^7}A_5+\frac{6\epsilon}{35a'}\,C=0,$$

$$(vi)\quad \frac{80\epsilon}{7a'^4}A_2-\frac{120}{a'^6}\left(1-\frac{20}{13}\epsilon\right)A_4+\frac{8!.21\epsilon}{65a'^8}\,A_6=0,$$

$$(vii)\quad \frac{600\epsilon}{11a'^5}A_3-\frac{144}{a'^7}\left(5-\frac{98}{11}\epsilon\right)A_5+\frac{9!.21\epsilon}{85a'^9}A_7=0, \qquad \ldots \quad (21)$$

etc.

We have taken ϵ to be very small and retained only its first power; so we put for A_1 in (i) its value for the case of the sphere, which is $-\frac{Ua'^3}{4}$.

Thus from (i) we have

$$C=\frac{3Ua'}{2}\left(1-\frac{1}{5}\epsilon\right) \qquad \ldots \qquad \ldots \qquad (22)$$

We have A_2, A_3,......all zero for the case of the sphere, therefore for the case of the spheroid the values of A_2, A_3,......will contain ϵ as a factor; therefore we neglect all terms like ϵA_2, ϵA_3,...as we take into account only the first power of ϵ.

Thus from (ii) we have

$$A_0=\frac{3va'}{2}\left(1-\frac{1}{5}\epsilon\right). \qquad \ldots \qquad \ldots \qquad (23)$$

Similarly from (iii) we get

$$A_1=-\frac{Ua'^3}{4}\left(1-\frac{57}{35}\epsilon\right). \qquad \ldots \qquad \ldots \qquad (24)$$

From (vi), we get

$$A_2=0.$$

Therefore from the equations it is clear that all the constants such as A_4, A_6,......(*i.e.* the A's with even suffixes) are all zero.

From (v) we have

$$A_3=-\frac{3Ua'^5}{140}\epsilon. \qquad \ldots \qquad \ldots \qquad (25)$$

A_5, A_7,...will contain ϵ^2, ϵ^3,...as factors and consequently they are neglected.

Thus finally after a slight simplification we find

$$u=\frac{Ua'}{4}\left[-4(1-\tfrac{1}{5}\epsilon)\frac{1}{r}-r^2(1-\tfrac{1}{5}\epsilon)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.$$
$$\left.+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{3}{r}\right)+\frac{3}{35}a'^4\epsilon\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right],$$

$$v=\frac{Ua'}{4}\left[-r^2(1-\tfrac{1}{5}\epsilon)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)+a'^2(1-\frac{57}{35}\epsilon)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)\right.$$
$$\left.+\frac{3}{35}a'^4\epsilon\frac{\partial^4}{\partial x^3\partial y}\left(\frac{1}{r}\right)\right],$$

$$w=\frac{Ua'}{4}\left[-r^2(1-\tfrac{1}{5}\epsilon)\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right)\right.$$
$$\left.+\frac{3}{35}a'^4\epsilon\frac{\partial^4}{\partial x^3\partial z}\left(\frac{1}{r}\right)\right].$$

etc.

Oberbeck's solution.

4. If an ellipsoid moves in an infinite mass of viscous liquid with the general velocity U, paralled to the axis of x such that $u=0$, $v=0$. $w=0$ at the surface and $u=\mathrm{U}$, $v=0$, $w=0$ at infinity, then Oberbeck has found out the following results for the motion of the liquid

$$u=\mathrm{U}+\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial x}-\mathrm{Q}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x^2}\right],$$

$$v=\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial y}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x\partial y}\right],$$

$$w=\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial z}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x\partial z}\right],$$

where

$$\mathrm{P}=-\pi abc\int_\lambda^\infty \frac{\dfrac{x^2}{a^2+s}+\dfrac{y^2}{b^2+s}+\dfrac{z^2}{c^2+s}-1}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}\,ds,$$

and

$$\mathrm{Q}=2\pi abc\int_\lambda^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}},$$

λ being the positive root of the equation

$$\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{c^2+s}=1,$$

and λ' and μ' are constants such that

$$\lambda'=\frac{\mathrm{U}}{\mathrm{Q}_0+a^2\mathrm{A}_0},\ \text{and}\ \mu'=a^2,$$

where

$$\mathrm{Q}_0=2\pi abc\int_0^\infty \frac{ds}{\sqrt{(a^2+s)\,(b^2+s)\,(c^2+s)}},$$

and

$$\mathrm{A}_0=2\pi abc\int_0^\infty \frac{ds}{(a^2+s)\ \sqrt{(a_2+s)\,(b^2+s)\,(c^2+s)}}.$$

I propose now to deduce, from these results given by Oberbeck, the values of u, v, w for the case, when $b=c$.

I take the ellipsoid of revolution to be ovary (*i.e.*, $b=c$) and its equation to be of the forms $r=a'[1+\epsilon P_2(\cos\theta)]$ where ϵ is very small, so that we may neglect its square and higher powers.

Therefore

$$a=a'\ (1+\epsilon),\ b=a'\ \left(1-\frac{\epsilon}{2}\right),\ a^2-b^2=a^2e^2,\ \text{and}\ \epsilon=\frac{e^2}{3}.$$

We have

P^* = the potential due to the spheroid at an external point

$$=\frac{3M}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{1}{1\cdot 3}\frac{(a^2-b^2)^{\frac{1}{2}}}{r}+\frac{1}{3\cdot 5}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)+\ \dots\right],$$

where $\qquad M=\dfrac{4\pi ab^2}{3}$, and $\cos\theta=\dfrac{x}{r}$.

Similarly

$$Q^*=\frac{M'}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{(a^2-b^2)^{\frac{1}{2}}}{r}+\frac{1}{3}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)+\ \dots\right],$$

where $\qquad M'=4\pi ab^2$.

Therefore

$$P=\frac{4\pi ab^2}{3}\cdot\frac{1}{r}+\frac{4\pi ab^2}{3}\frac{(a^2-b^2)}{2\cdot 5}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \dots\ \dots,$$

and

$$Q=\frac{4\pi ab^2}{3}\cdot\frac{3}{r}+\frac{4\pi ab^2}{3}\frac{(a^2-b^2)}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \dots\ \dots.$$

Substituting these values of P and Q, we have

$$u=U+\frac{U}{Q_0+a^2A_0}\frac{4\pi ab^2}{3}\left[3x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{a^2e^2}{2}x\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right.$$

$$-\frac{3}{r}-\frac{a^2e^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+a^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)$$

$$\left.+\frac{a^4e^2}{10}\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right],$$

neglecting higher powers of e^2.

* See Byerly's, *Spherical Harmonics*, pp. 155 and 156.

We have

$$x=\frac{P_1}{r},\ P_1{}^2=\frac{P_0+2P_2}{3},\ \text{and}\ P_1P_3=\frac{4P_4+3P_2}{7}.$$

Therefore

$$u=U+\frac{U}{Q_0+a^2A_0}\frac{4\pi ab^2}{3}\left[-\frac{4P_0}{r}-\frac{2P_2}{r}+a^2\left(1-\frac{8}{7}e^2\right)\frac{2P_2}{r^3}\right.$$
$$\left.-\frac{12}{7}a^2e^2\frac{P_4}{r^3}+\frac{4\,!}{10}a^2e^2\frac{P_4}{r^5}\right].$$

i.e.,

$$u=U+\frac{U}{Q_0+a^2A_0}\frac{4\pi a'^3}{3}\left[-\frac{4P_0}{r}-\frac{2P_2}{r}+a'^2\left(1-\frac{10}{7}\epsilon\right)\frac{2P_2}{r^3}\right.$$
$$\left.-\frac{3}{14}a'^2\epsilon\frac{4\,!\,P_4}{r^3}+\frac{3}{10}a'^4\epsilon\frac{4\,!\,P_4}{r^5}\right].$$

From the boundary conditions we have

$$u=0,\ \text{when}\ r=a'(1+\epsilon P_2)$$

Therefore

$$0=U+\frac{U}{Q_0+a^2A_0}\frac{4\pi a'^3}{3}\left[-\frac{4P_0}{a'}(1-\epsilon P_2)-\frac{2P_2}{a'}(1-\epsilon P_2)\right.$$
$$+a'^2\left(1-\frac{10}{7}\epsilon\right)\frac{2P_2}{a'^3}(1-3\epsilon P_2)$$
$$\left.-\frac{3}{14}a'^2\epsilon\frac{4\,!\,P_4}{a'^3}+\frac{3}{10}a'^4\epsilon\frac{4\,!\,P_4}{a'^5}\right],$$

negleding square and higher powers of ϵ.

From this equation comparing the coefficients of P_0, we have

$$U+\frac{U}{Q_0+a^2A_0}\frac{4\pi a'^3}{3}\left[-\frac{4}{a'}+\frac{2\epsilon}{5a'}-\frac{6\epsilon}{5a'}\right]=0,$$

since $$P_2=\frac{18}{35}P_4+\frac{2}{7}P_2+\frac{1}{7}P_0.$$

From this we find

$$Q_0+a^2A_0=\frac{4\pi a'^3}{3}\left(4+\frac{4}{5}\epsilon\right).$$

Thus finally we find

$$u=U+\frac{Ua'}{4}\left[-4\left(1-\frac{1}{5}\epsilon\right)\frac{1}{r}-r^2\left(1-\frac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.$$
$$+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)$$
$$\left.+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right].$$

Similaryly we can find out the values of v and w in terms of the differential coefficients of $\frac{1}{r}$; as

$$v=\frac{Ua'}{3}\left[-r^2\left(1-\frac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)\right.$$
$$\left.-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^3\,\partial y}\left(\frac{1}{r}\right)+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^3\,\partial y}\left(\frac{1}{r}\right)\right],$$

$$w=\frac{Ua'}{4}\left[-r^2\left(1-\frac{1}{5}\epsilon\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right)\right.$$
$$\left.-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^3\,\partial z}\left(\frac{1}{r}\right)+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^3\,\partial z}\left(\frac{1}{r}\right)\right].$$

Resistance

5. Next I propose to find out an expression for the resistance experienced by the ellipsoid in moving through the liquid.

Let F denote the resistance; then for the case of the ellipsoid of three unequal axes, Oberbeck has found out that

$$\text{the resistance}=6\mu\pi RU, \text{ and } R=\frac{8}{3}\,\frac{abc}{X_0+a^2\alpha_0}$$

where

$$X_0=abc\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\text{ and}$$

$$\alpha_0=abc\int_0^\infty\frac{d\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

In the present case $b=c$, $a=a'(1+\epsilon)$, $b=a'\left(1-\frac{\epsilon}{2}\right)$

and $$X_0+a^2\alpha_0=\frac{2}{3}a'^2\left(4+\frac{4}{5}\epsilon\right).$$

Therefore $$R=\frac{8}{3}\,\frac{a'^3}{\frac{2}{3}a'^2\left(4+\frac{4}{5}\epsilon\right)}=\frac{a'}{1+\frac{1}{5}\epsilon}$$

and consequently $$F=6\pi\mu a'U\left(1-\frac{1}{5}\epsilon\right);$$

and from my results I get the same expression for the resistance.

6. The method used by me for the ellipsoid of revolution is capable of being extended to the case of the ellipsoid of three unequal axes.

On a class of ellipsoidal harmonics and a method of solving the wave equation in ellipsoidal coordinates.

BY

SUDHANSUKUMAR BANERJI.

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1. In the present p. per I have developed a new class of ellipsoidal harmonics which are solutions of Laplace's equation and then have used these harmonics in solving the wave equation in ellipsoidal coordinates. The ellipsoidal coordinates used in this paper are (ρ, θ, ϕ) defined by

$$\begin{aligned} x &= a\rho \sin\theta \cos\phi, \\ y &= b\rho \sin\theta \sin\phi, \qquad \dots \quad (1) \\ z &= c\rho \cos\theta, \end{aligned}$$

where ρ=constant obviously determines a set of similar and similarly situated ellipsoids. This system being analogous to the ordinary polar coordinates has got certain advantage over the more usual system λ, μ, ν representing a set of confocal ellipsoidal surfaces, hyperboloids of one sheet and hyperboloids of two sheets respectively but has also got certain disadvantage in as much as it does not form an orthogonal system. The ellipsoidal harmonics in the coordinates (ρ, θ, ϕ) developed in this paper will be found to be simpler and more convenient for application to physical problems than the Lame's functions. But perhaps the most remarkable application of these harmonics consists in the use that has been made of them in this paper in solving the wave equation in these coordinates. This equation which was first transformed by Mathieu[1] in λ, μ, ν was found to be so unmanageable that he had to content himself with approximating to its solution for the special case of an ellipsoid of revolution. Subsequent writers including Prof. Niven[2] have simply improved upon the approximations of Mathieu.

[1] *Cours de Physique Mathematique*, Ch. IX.

[2] *Phil. Trans.*, Vol. CLXXI, (1880).

2. It is well-known that if (r, θ, ϕ) denote the spherical polar coordinates of a point (x, y, z),

$$r^n \, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi$$

$$=\frac{(n+m)(n+m-1)\dots(n+1)}{2\pi}(-1)^{\frac{m}{2}}\int_0^{2\pi}(z+ix\cos u+iy\sin u)^n \begin{matrix}\cos\\ \sin\end{matrix} mu\,du. \quad \dots \quad (2)$$

Obviously, by a generalisation of this expression we can define a function $C_n^m(\theta, \phi)$ by the expression

$$C_n^m(\theta, \phi)=\frac{(n+m)(n+m-1)\dots(n+1)}{2\pi}(-1)^{\frac{m}{2}}$$

$$\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^n\cos mu\,du \quad \dots \quad (3)$$

With this definition for the function $C_n^m(\theta, \phi)$, it is easy to see that $\rho^n\,C_n^m(\theta, \phi)$ is a solution of Laplace's equation in the ellipsoidal coordinates (ρ, θ, ϕ) defined by (1).

Similarly we can define a function $S_n^m(\theta, \phi)$ by the expression

$$S_n^m(\theta, \phi)=\frac{(n+m)(n+m-1)\dots(n+1)}{2\pi}(-1)^{\frac{m}{2}}$$

$$\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^n\sin mu\,du \quad \dots \quad (4)$$

and $\rho^n\,S_n^m(\theta, \phi)$ is another solution of Laplace's equation in (ρ, θ, ϕ).

The function corresponding to the Legendre's function can be defined by

$$C_n(\theta, \phi)=\frac{1}{2\pi}\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^n du$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[c\cos\theta+i(a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi)^{\frac{1}{2}}\cos u\right]^n du \quad \dots \quad (5)$$

and $\rho^n C_n(\theta, \phi)$ is a solution of Laplace's equation in (ρ, θ, ϕ). When $a=b$, the function is independent of ϕ, *i.e.*,

$$C_n(\theta)=\frac{1}{2\pi}\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos u)^n\,du.$$

3. To obtain the harmonics which vanish at infinity we define the functions

$$\mathfrak{C}_n^m(\theta,\phi)=(-1)^m\frac{n(n-1)\ \dots\ (n-m+1)}{2\pi}$$

$$\int_0^{2\pi}\frac{\cos mu\,du}{(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^{n+1}}\quad\dots\quad(6)$$

and

$$\mathfrak{S}_n^m(\theta,\phi)=(-1)^m\frac{n(n-1)\ \dots\ (n-m+1)}{2\pi}$$

$$\int_0^{2\pi}\frac{\sin mu\,du}{(c\cos\theta+ia\sin\theta\cos\phi\cos u+ib\sin\theta\sin\phi\sin u)^{n+1}}\quad\dots\quad(7)$$

and obviously $\mathfrak{C}_n^m(\theta,\phi)/\rho^{n+1}$ and $\mathfrak{S}_n^m(\theta,\phi)/\rho^{n+1}$ are solutions of Laplace's equation which vanish at infinity.

4. By an application of Green's theorem we can easily prove that the functions $C_n^m(\theta,\phi)$, $S_n^m(\theta,\phi)$, $\mathfrak{C}_n^m(\theta,\phi)$ and $\mathfrak{S}_n^m(\theta,\phi)$ defined above all satisfy conjugate properties.

The element of volume in the co-ordinates (ρ, θ, ϕ) is

$$abc\,\rho^2\sin\theta\,d\rho\,d\theta\,d\phi\quad\dots\quad(8)$$

which can also be written in the form

$$dp.dS,\quad\dots\quad(9)$$

where dS is a surface element and

$$dp=p_0 d\rho.$$

$$p_0=\frac{abc}{(b^2c^2\sin^2\theta\cos^2\phi+c^2a^2\sin^2\theta\sin^2\phi+a^2b^2\cos^2\theta)^{\frac{1}{2}}}.\quad\dots\quad(10)$$

So that the surface element dS can be written in the form

$$dS = \frac{abc}{p_0}\, \rho^2 \sin\theta\, d\theta\, d\phi. \qquad \ldots \quad (11)$$

Now, by Green's theorem, if Φ and Φ' be two functions which satisfy Laplace's equation, we have

$$\iint \left(\Phi' \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \Phi'}{\partial n}\right) dS = 0. \qquad \ldots \quad (12)$$

Hence since $\dfrac{\partial}{\partial n} = \dfrac{1}{p_0}\dfrac{\partial}{\partial \rho}$, we see at once that the functions $C_n^m(\theta,\phi)$ and $S_n^m(\theta,\phi)$ defined above satisfy the following conjugate properties :—

$$\int_0^\pi \int_0^{2\pi} C_n^m(\theta,\phi) C_{n'}^m(\theta,\phi) \frac{\sin\theta}{p_0^2}\, d\theta d\phi = 0, \qquad (n \neq n') \ldots \quad (13)$$

$$\int_0^\pi \int_0^{2\pi} S_n^m(\theta,\phi)\, S_{n'}^m(\theta,\phi) \frac{\sin\theta}{p_0^2}\, d\theta\, d\phi = 0, \qquad (n \neq n') \ldots \quad (14)$$

Also

$$\int_0^\pi \int_0^{2\pi} [C_n^m(\theta,\phi)]^2 \frac{\sin\theta}{p_0^2}\, d\theta\, d\phi = \text{const.}, \qquad \ldots \quad (15)$$

$$\int_0^\pi \int_0^{2\pi} [S_n^m(\theta,\phi)]^2 \frac{\sin\theta}{p_0^2}\, d\theta d\phi = \text{const.} \qquad \ldots \quad (16)$$

Similarly for the functions $\mathfrak{C}_n^m(\theta,\phi)$, $\mathfrak{S}_n^m(\theta,\phi)$.

5. It is interesting to note the following relations between the functions $C_n^m(\theta,\phi)$, $S_n^m(\theta,\phi)$, $\mathfrak{C}_n^m(\theta,\phi)$ and $\mathfrak{S}_n^m(\theta,\phi)$:—

$$C_n^m(\theta,\phi) = (a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta)^{\frac{2n+1}{2}}\, \mathfrak{C}_n^m(\theta,\phi), \qquad (17)$$

$$S_n^m(\theta,\phi) = (a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta)^{\frac{2n+1}{2}}\, \mathfrak{S}_n^m(\theta,\phi). \qquad (18)$$

When $a=b=c=1$, $C_n^m(\theta,\phi) = \mathfrak{C}_n^m(\theta,\phi) = P_n^m(\cos\theta)\cos m\phi$

and $S_n^m(\theta,\phi) = \mathfrak{S}_n^m(\theta,\phi) = P_n^m(\cos\theta)\sin m\phi$.

The values of the functions $C_n^m(\theta,\phi)$, etc., can be expressed in terms of the hypergeometric function.

Thus we get

$$C_n^m(\theta,\phi)=\frac{(n+m)!}{(n-m)!}\,\frac{i^m R^{2n+1} r^m \tan^m\theta}{2^m m!\,(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2},\frac{n+m+2}{2},m+1,-r^2\tan^2\theta\right)\cos m\psi$$

$$S_n^m(\theta,\phi)=\frac{(n+m)!}{(n-m)!}\,\frac{i^m R^{2n+1} r^m \tan^m\theta}{2^m m!\,(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2},\frac{n+m+2}{2},m+1,-r^2\tan^2\theta\right)\sin m\psi$$

where F is a hypergeometric function of the four elements within the parenthesis and

$$R^2=a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi+c^2\cos^2\theta,$$

$$r^2=(a^2\cos^2\phi+b^2\sin^2\phi)/c^2,$$

$$\tan\psi=b/a\tan\phi,$$

c being the greatest axis of the ellipsoid.

5. It is well-known that

$$P_n(\sin\theta\cos\phi\sin u\cos v+\sin\theta\sin\phi\sin u\sin v+\cos\theta\cos u)$$

$$=P_n(\cos\theta)\,P_n(\cos u)+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta)P_n^m(\cos u)\cos m(\phi-v)$$

If we write

$$x=r\sin\theta\cos\phi,\qquad x'=r'\sin u\cos v,$$

$$y=r\sin\theta\sin\phi,\qquad y'=r'\sin u\sin v,$$

$$z=r\cos\theta,\qquad z'=r'\cos u.$$

and $$\Pi_{n,m}=\frac{(n+m)(n+m-1)\dots(n+1)}{2\pi}(-1)^{\frac{m}{2}},$$

then we can write the above identity in the form

$$r^n r'^n\,P_n(\sin\theta\cos\phi\sin u\cos v+\sin\theta\sin\phi\sin u.\sin v+\cos\theta\cos u)$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n d\omega\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n d\omega$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\Pi^2_{n,m}\left[\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\cos m\omega d\omega\right.$$

$$\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\cos m\omega d\omega$$

$$+\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\sin m\omega d\omega$$

$$\left.\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega\right],$$

that is to say in the form

$$\frac{1}{2\pi}\int_0^{2\pi}[xx'+yy'+zz'+i\{(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2\}^{\frac{1}{2}}\cos\omega]^n d\omega$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n d\omega\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n d\omega$$

$$+\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\Pi^2_{n,m}\left[\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\cos m\omega d\omega\right.$$

$$\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\cos m\omega d\omega$$

$$+\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega$$

$$\left.\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega\right].$$

Now if we write

$$x=a\rho\sin\theta\cos\phi,\qquad x'=r'\sin u\cos v,$$

$$y=b\rho\sin\theta\sin\phi,\qquad y'=r'\sin u\sin v,$$

$$z=c\rho\cos\theta,\qquad z'=r'\cos u,$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi}\Big[a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u$$

$$+i\{(b\sin\theta\sin\phi\cos u-c\sin u\sin v\cos\theta)^2$$

$$+(c\sin u\cos v\cos\theta-a\sin\theta\cos\phi\cos u)^2$$

$$+(a\sin\theta\cos\phi\sin u\sin v-b\sin\theta\sin\phi\sin u\cos v)^2\}^{\frac{1}{2}}\cos\omega\Big]^n d\omega$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\,d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\,d\omega$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!\;\Pi^2}{(n+m)!\,n,m}\Bigg[\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\cos m\omega d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\cos m\omega d\omega$$

$$+\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\sin m\omega d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\sin m\omega d\omega\Bigg].$$

In other words, we get in accordance to our previous definition

$$C_n(\theta,\phi\,;u,v)=C_n(\theta,\phi)\,P_n(\cos u)$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\Big[C_n^m(\theta,\phi)\,P_n^m(\cos u)\cos mv+S_n^m(\theta,\phi)\,P_n^m(\cos u)\sin mv\Big]\quad\ldots\quad(19)$$

7. We shall now obtain the solution of the wave equation

$$(\nabla^2+k^2)V=0\quad\ldots\quad(20)$$

in terms of the functions introduced in the previous articles.

It is well-known that

$$V=\int_0^{\pi}\int_0^{2\pi} e^{ik(x\sin u\cos v+y\sin u\sin v+z\cos u)} f(u,v)\,du\,dv \quad \ldots \quad (21)$$

represents a solution of the above equation in cartesian coordinates.

Hence in the coordinates (ρ, θ, ϕ), a solution of the above equation will be given by

$$V=\int_0^{\pi}\int_0^{2\pi} e^{ik\rho(a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u)} f(u, v)\,du\,dv \ \ldots \quad (22)$$

It is obvious that by properly choosing the function $f(u, v)$ we can construct a set of solutions of the equation $(\nabla^2+k^2)V=0$ in ρ, θ, ϕ.

Let us define the function $\psi_n(k\rho)$ by the relation

$$\psi_n(k\rho)=\frac{i^{-n}}{2\pi}\int_0^{\pi}\int_0^{2\pi} e^{ik\rho c\cos\theta}\, C_n(\theta, \phi)\, \frac{\sin\theta}{p_0^{\,2}}\, d\theta d\phi. \quad \ldots \quad (23)$$

When $a=b$,

$$\psi_n(k\rho)=\frac{i^{-n}}{b^2c^2}\int_0^{\pi} e^{ik\rho c\cos\theta}\, C_n(\theta)(c^2\sin^2\theta+a^2\cos^2\theta)\sin\theta d\theta d\phi. \ \ldots \quad (24)$$

By expanding $e^{ik\rho c\cos\theta}$ in the exponential series, it is easy to evaluate the integral term by term and to obtain an expression for $\psi_n(k\rho)$ in a series of ascending powers of $k\rho$.

One way[1] of expressing the result is

$$\psi_n(k\rho)=\sum_s A_s(ck\rho)^s,$$

[1] See a note by Prof. Baker on a formula connected with the theory of spherical harmonics, *Proc. Lond. Math. Soc.*, Vol. XV, (1916).

where the summation extends for all values of s for which $n+s+2$ is an even integer and

$$A_s = \frac{i^{s-n}}{4\pi^2 a^2 b^2 c^2} \left[\frac{2^s \partial^s}{\partial \zeta^s} U_s + 2^{s-2} \frac{s(s-1)}{1!} \frac{\partial^{s-2}}{\partial \zeta^{s-2}} U_{s-2} \right.$$

$$\left. + 2^{s-4} \frac{s(s-1)(s-2)(s-3)}{2!} \frac{\partial^{s-4}}{\partial \zeta^{s-4}} U_{s-4} + \dots \right].$$

$$U_q = \frac{1}{[\frac{1}{2}(n+2-q)]!} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \right]^{\frac{1}{2}(n+2-q)} U,$$

$$U = (b^2c^2\xi^2 + c^2a^2\eta^2 + a^2b^2\zeta^2) \int_0^{2\pi} (c\zeta + ia\xi \cos\omega + ib\eta \sin\omega)^n d\omega$$

and ξ, η, ζ stand for $\sin\theta \cos\phi$, $\sin\theta \sin\phi$ and $\cos\theta$.

With this definition for $\psi_n(k\rho)$, it is obvious that we can expand $e^{ik\rho c \cos\theta}$ in a series of the type

$$e^{ik\rho c\cos\theta} = A_0\psi_0(k\rho) + A_1\psi_1(k\rho)C_1(\theta, \phi) + A_2\psi_2(k\rho)C_2(\theta, \phi) + \dots$$

$$+ A_n\psi_n(k\rho)C_n(\theta, \phi) + \dots, \quad \dots \quad (25)$$

where A_n's are simple numerical constants and are given by

$$A_n = 2\pi i^n \Bigg/ \int_0^{\pi} \int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_0^2} \, d\theta d\phi. \quad \dots \quad (26)$$

Similarly, we obtain the expansion

$$e^{ik\rho(a \sin\theta \cos\phi \sin u \cos v + b \sin\theta \sin\phi \sin u \sin v + c \cos\theta \cos u)}$$

$$= A_0\psi_0(k\rho) + A_1\psi_1(k\rho)C_1(\theta, \phi; u, v) + \dots$$

$$+ A_n\psi_n(k\rho)C_n(\theta, \phi; u, v) + \dots, \quad \dots \quad (27)$$

the constants A_0, A_1, etc. having the same values as before.

The expression for V can therefore be written in the form

$$V = \sum_{n=0}^{\infty} A_n \psi_n(k\rho) \int_0^{\pi} \int_0^{2\pi} C_n(\theta, \phi; u, v) f(u, v) du dv. \quad \dots \quad (28)$$

If now $C_n(\theta, \phi; u, v)$ be expanded in a series of the form (19) and if $f(u, v)$ be chosen to be

$$\sin u \; P_n^m(\cos u) \cos mv,$$

a solution of the wave equation in the co-ordinates ρ, θ, ϕ is at once obtained in the form

$$\psi_n(k\rho) C_n^m(\theta, \phi). \quad \dots \quad (29)$$

If on the other hand $f(u, v)$ is taken to be

$$\sin u \; P_n^m(\cos u) \sin mv,$$

another solution of the wave equation is obtained in the form

$$\psi_n(k\rho) S_n^m(\theta, \phi). \quad \dots \quad (30)$$

A number of interesting physical problems can be solved with the help of the solutions obtained above. For example, the non-stationary state of heat in an ellipsoid given by $\rho=1$, with the condition of zero temperature at the boundary, can be expressed in terms of (29) and (30), k being a root of the equation

$$\psi_n(k)=0.$$

Similarly, the periods of free oscillations of a gas, contained within the ellipsoidal shell $\rho=1$, are given by k which are the roots of the equation

$$\psi_n'(k)=0.$$

A memoir by the present writer on the many elegant and interesting properties of the functions introduced in this paper and their applications to physical problems involving ellipsoidal boundaries will be published shortly.

Some cases of Tidal oscillations in canals of variable section.

BY

SASADHAR DASGUPTA.

[Read March 9th, 1919.]

1. Problems on seiches in lakes and tidal waves in estuaries have attracted considerable attention from mathematicians for a long time. Prof. Chrystal[1] and Lamb[2] have attempted to give a satisfactory mathematical theory of these phenomena. In view of the very interesting theoretical results obtained by these writers, I was led to study some more cases not considered by them.

Towards the end of the paper I have considered the second and higher order waves in a parabolic lake.

It may be remarked that the only case for which the second order tides have been determined is the one considered by Airy[3] and Mc. Cown[4] in which the section is uniformly rectangular throughout. As usual I find that the frequency of the "second order tide" is double that of the primary disturbance.

I am thankful to Dr. S. K. Banerji for the interest he has taken in the preparation of this paper.

PART I.

First Order Tides.

2. The free tidal oscillations in canals of variable section are determined by the equations

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b(x)} \frac{\partial}{\partial x} \left[S. \frac{\partial \eta}{\partial x} \right], \qquad \dots \quad (1)$$

$$\frac{\partial^2}{\partial t^2} [S\xi] = gS. \frac{\partial}{\partial x} \left[\frac{1}{b(x)} \cdot \frac{\partial}{\partial x} (S\xi) \right], \qquad \dots \quad (2)$$

$$\eta = -\frac{1}{b(x)} \frac{\partial}{\partial x} (S\xi); \qquad \dots \quad (3)$$

[1] Chrystal, "Some results in the mathematical theory of seiches" *Trans R. S. Edin.*, t. XLI, p. 599 (1905).

[2] Lamb's *Hydrodynamics*, 4th edition, p. 267.

[3] Airy, "Tides and Waves," *Ency. Metrop.*, Art. 192, (1845).

[4] Mc. Cown, "On the theory of long waves" *Phil. Mag.*, (5), t. XXXV, p. 250 (1893).

where S is the section of the canal at the point x, η the tidal elevation above the equilibrium level, $b(x)$ the breadth, and ξ the time integral of the displacement past the plane x up to the time t.

I now proceed to obtain the solution of these equations for various types of canals.

CASE I.

3. Suppose that the horizontal section of the canal is a parabola given by $b(x)=\frac{x^2}{2c}$ and that the depth is constant.

Assuming that $\eta \propto \cos(\sigma t+\epsilon)$ we see from (1) that

$$\frac{d^2\eta}{dx^2}+\frac{2}{x}\frac{d\eta}{dx}+k^2\eta=0 \qquad \dots \quad (4)$$

where $k^2=\sigma^2/gh$.

The solution of (4) is given by

$$\eta=\mathrm{A}x^{-\frac{1}{2}}\mathrm{J}_{\frac{1}{2}}(kx)\cos(\sigma t+\epsilon)$$

$$=\mathrm{A}x^{-\frac{1}{2}}\left(\frac{2}{\pi kx}\right)^{\frac{1}{2}}\sin(kx).\ \cos(\sigma t+\epsilon).$$

If the canal communicates with an open sea at its mouth $x=a$ at which tidal oscillations of the type $\eta=\mathrm{C}\cos(\sigma t+\epsilon)$ are maintained, then

$$\eta=\mathrm{C}\frac{a}{x}\frac{\sin(kx)}{\sin(ka)}\cos(\sigma t+\epsilon).$$

If the canal be closed at $x=a$, the admissible values of k are given by

$$\left|\frac{\partial\eta}{\partial x}\right|_{x=a}=0,$$

$$i.e.,\ \frac{\partial}{\partial x}\left[x^{-\frac{1}{2}}\mathrm{J}_{\frac{1}{2}}(kx)\right]_{x=a}=0,$$

or, $$\tan(ka)=ka.$$

4. The depth being constant, let us inquire under what circumstances,

$$\frac{\partial^2\eta}{\partial t^2}=\frac{gh_0}{b(x)}\frac{\partial}{\partial x}\left(b(x)\frac{\partial\eta}{\partial x}\right) \qquad \dots \quad (5)$$

has a solution of the form

$$\eta=\phi(x)\mathrm{F}\left[\psi(x)\pm\sigma t\right]. \qquad \dots \quad (6)$$

where F is an arbitrary function and ϕ, ψ are definite functions of the argument x.

Now (5) can be written in the form

$$\frac{1}{gh}\frac{\partial^2}{\partial t^2}=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\log\left[b(x)\right]}{\partial x} \qquad \dots \quad (7)$$

Substituting the value of η from (6) in (7) we have

$$\mathrm{F}''\left(\frac{\sigma^2}{gh}\phi-\phi\psi'^2\right)-\mathrm{F}'\left(2\phi'\psi'+\phi\psi''+\frac{\partial\log[b(x)]}{\partial x}\cdot\phi\psi'\right)$$
$$-\mathrm{F}\left(\phi''+\frac{\partial\log\left[b(x)\right]}{\partial x}\phi'\right)=0.$$

Since F is arbitrary :—

$$\psi'^2-\frac{\sigma^2}{gh}=0,$$

$$2\phi'\psi'+\phi\psi''+\frac{\partial\log\left[b(x)\right]}{\partial x}\cdot\phi\psi'=0,$$

$$\phi''+\frac{\partial\log b(x)}{\partial x}\phi'.=0.$$

$$\therefore \quad \psi(x)=\pm\frac{\sigma x}{(gh)^{\frac{1}{2}}}+\mathrm{C}_1,$$

$$\phi(x)=(\mathrm{B}x+\mathrm{C})^{-1},$$

$$b(x)=\mathrm{A}\;(\mathrm{B}x+\mathrm{C})^2.$$

If $\mathrm{C}_1=\mathrm{C}=0$ and also $\frac{\sigma^2}{gh}=k^2$, then assuming $\mathrm{F}=e^{i(kx\pm\sigma t)}$ we get the harmonic solution which we have already considered.

CASE II.

5. Assuming that both the bed and surface of the canal are sloping, so that $S=b_0 h_0 x^2$, the other circumstances being the same as in case I, we get from equation (1),

$$\frac{1}{x}\frac{\partial}{\partial x}\left[x^2\frac{\partial \eta}{\partial x}\right]+\frac{\sigma^2}{gh_0}\eta=0.$$

Putting $z^2=2x$ and $\frac{2\sigma^2}{gh_0}=k$, we have

$$\frac{d^2\eta}{dz^2}+\frac{3}{z}\frac{d\eta}{dz}+k\eta=0.$$

Therefore, $\eta = Az^{-1} J_1 \left(k^{\frac{1}{2}}z\right) \cos(\sigma t+\epsilon)$

$$=A(2x)^{-\frac{1}{2}} J_1\left[(2kx)^{\frac{1}{2}}\right] \cos(\sigma t+\epsilon).$$

On determining the constant, we have

$$\eta = B\left(\frac{x}{a}\right)^{-\frac{1}{2}}\frac{J_1\left[(2kx)^{\frac{1}{2}}\right]}{J_1\left[(2ka)^{\frac{1}{2}}\right]}\cos(\sigma t+\epsilon).$$

Now if we assume that $\eta \propto \cos(n\sigma t)$ where η is arbitrary and $\sigma^2=\frac{gh_0}{2}$, a particular integral of

$$\frac{1}{x}\frac{\partial}{\partial x}\left[x^2\frac{\partial \eta}{\partial x}\right]+\frac{n^2}{2}\eta=0 \text{ is}$$

$$\eta=A\left[(2x)^{\frac{1}{2}}\right]^{-1} J_1\left[n(2x)^{\frac{1}{2}}\right]\cos n\sigma t.$$

$$=-\frac{A}{n}\frac{\partial J_0\left[n(2x)^{\frac{1}{2}}\right]}{\partial x}.\cos n\sigma t.$$

$$=-\frac{A}{n\pi}\frac{\partial}{\partial x}\int_0^\pi \cos\left[n(2x)^{\frac{1}{2}}\cos\phi\right]\cos n\sigma t.\, d\phi$$

$$=-\frac{A}{2n\pi}\frac{\partial}{\partial x}\int_0^\pi\left[\cos n\{(2x)^{\frac{1}{2}}\cos\phi+\sigma t\}+\cos n\{(2x)^{\frac{1}{2}}\cos\phi-\sigma t\}\right]d\phi$$

Therefore we have as a solution :—

$$\eta = \frac{\partial}{\partial x}\int_0^\pi \left[F\{(2x)^{\frac{1}{2}}\cos\phi + \sigma t\} + F\{(2x)^{\frac{1}{2}}\cos\phi - \sigma t\} \right] d\phi$$

where F is a function capable of being expanded in a series of cosines.

To determine F we shall have to satisfy the initial condition. Suppose $\eta = \eta_0$ when $t=0$, then

$$\int \eta_0 \, dx = 2\int_0^\pi F[(2x)^{\frac{1}{2}} \cos\phi] d\phi.$$

If $\int \eta_0 dx = \eta_1$, then the determination of F involves the solution of the integral equation

$$\eta_1 = 2\int_0^\pi F\,[(2x)^{\frac{1}{2}} \cos\phi] d\phi.$$

CASE III.

6. If the surface is parabolic and the bed sloping, such that $h(x) = h_0 . x$, $[b.(x)]^2 = ax$, then the equation (1) becomes

$$\frac{1}{x^{\frac{1}{2}}}\frac{\partial}{\partial x}\left[x^{\frac{3}{2}}\frac{\partial \eta}{\partial x}\right] + \frac{\sigma^2}{gh_0}\eta = 0.$$

Putting $z^2 = 2x$, and $k^2 = \dfrac{2\sigma^2}{gh_0}$ we get

$$\frac{d^2\eta}{dz^2} + \frac{2}{z}\frac{d\eta}{dz} + k^2\eta = 0.$$

Since η is to be finite at the origin

$$\eta = Az^{-\frac{1}{2}} J_{\frac{1}{2}}(kz) \cos(\sigma t + \epsilon)$$

$$= A\left[(2x)^{\frac{1}{2}}\right]^{-\frac{1}{2}} J_{\frac{1}{2}}\,[k(2x)^{\frac{1}{2}}] \cos(\sigma t + \epsilon).$$

Assuming that the bed is sloping, let us enquire under what circumstances (1) may have a solution of the form :

$$\eta=\phi(x)\ F\ (\psi(x)\pm\sigma t).$$

Following the same method as in case I, we get

$$\phi(x)=(Bx^{\frac{1}{2}}+C)^{-1}$$

and $b(x)=\dfrac{D[Bx^{\frac{1}{2}}+C]^2}{x^{\frac{1}{2}}}$.

CASE IV.

7. Let us now consider the case[1] when the breadth is constant, the longitudinal section through the x axis is parabolic being given by $h\ (x)=h_0\left(1-\dfrac{x^2}{a^2}\right)$ and the section perpendicular to the axis at any point a parabola of latus rectum, $4k$ say, where k varies from point to point and as can be easily seen is given by

$$k=\frac{b_0{}^2}{4h_0\left(1-\dfrac{x^2}{a^2}\right)}.$$

When $x=\pm a$, the limiting parabolas at the extremities of the lake coincide with the bounding lines.

Putting $u=S.\xi,\ v=\displaystyle\int b(x)dx$ in (2) and (3) we have,

$$\frac{\partial^2 u}{\partial t^2}=g.\ S.\ b(x)\frac{\partial^2 u}{\partial v^2} \qquad \dots \quad (8)$$

$$\eta=-\frac{\partial u}{\partial v} \qquad \dots \quad (9)$$

where

$$S=\frac{4b_0h_0}{3}\left(1-\frac{x^2}{a^2}\right).$$

[1] Chrystal and Lamb have considered the section perpendicular to the x axis to be rectangular.

Substituting in (8) and assuming that $u \propto \cos(\sigma t+\epsilon)$ and putting $x=a\omega$, we have

$$(1-\omega^2)\,\frac{\partial^2 u}{\partial \omega^2}+cu=0, \qquad \text{where } c=\frac{3\sigma^2 a^2}{4gh_o} \qquad \ldots \quad (10)$$

This equation has the following solutions consistent with the boundary conditions :—

(a)
$$\xi=\frac{3A}{4b_o h_o}\,\frac{C(c_{2s-1},\omega)}{(1-\omega^2)}\cos(\sigma_{2s-1}t+\epsilon),$$

$$\eta=-\frac{1}{b_o a}\,\frac{\partial u}{\partial \omega}=-\frac{A}{ab_o}\,\frac{\partial}{\partial \omega}\left[C(c_{2s-1},\omega)\right]\cos(\sigma_{2s-1}t+\epsilon),$$

where $C(c,\omega)$ is Chrystal's seiche cosine function and c_{2s-1} is a root of $C(c,1)=0$, it being $=2s.\,(2s-1)$ where s is an integer.

(b)
$$\xi=\frac{3B.S(c_{2s},\omega)}{4b_o h_o(1-\omega^2)}\cos(\sigma_{2s}t+\epsilon),$$

$$\eta=-\frac{B}{ab_c}\,\frac{\partial}{\partial \omega}\left[S\,(c_{2s},\omega)\right]\cos(\sigma_{2s}t+\epsilon)$$

where $S\,(c,\omega)$ is Chrystal's seiche sine function and c_{2s} is a root of $S\,(c,1)=0$, it being $=2s(2s+1)$ where s is an integer.

In either case the period of the n—nodal seiche is given by

$$T_n=\frac{2\pi}{\sigma_n}=\frac{\pi l}{\sqrt{n(n+1)gh_o}}\,\frac{\sqrt{3}}{2}\ , \text{ where } l=\text{length of the lake.}$$

This shows that the period of the n—nodal seiche is $\frac{\sqrt{3}}{2}$ times the corresponding period for a lake with tarnsverse rectangular section.

N.B.—If the longitudinal section be a convex parabola, the other circumstances being the same as in case IV we get the solution in terms of Chrystal's seiche hyperbolic sine and cosine functions. The period is however given by a similar expression.

CASE V.

8. If $b(x)=b_o$, $h(x)=h_o\sqrt{1-\frac{x^2}{a^2}}$, *i.e.*, if the longitudinal section is an ellipse whose major axis is in the equilibrium level, we have

$$\frac{\partial}{\partial x}\left[\sqrt{1-\frac{x^2}{a^2}}\frac{\partial\eta}{\partial x}\right]+\frac{\sigma^2}{gh_o}\eta=0 \qquad \ldots \quad (11)$$

Putting $\frac{x}{a}=\sin 2z$, we get

$$\frac{d^2\eta}{dz^2}+2\theta_1\cos 2z.\eta=0, \text{ where } \theta_1=\frac{2\sigma^2a^2}{gh_o}.$$

To solve it assume $\eta=e^{\mu z}\sum_{n=-\infty}^{n=\infty} b_n e^{2niz}$

Substituting in (1) and equating the co-efficients of powers of e^{2iz} to zero, we obtain the system of equations

$$(\mu+2ni)^2b_n+\theta_1b_{n-1}+\theta_1b_{n+1}=0,$$

$$n=[\ldots-2,-1,0,1,2\ldots]$$

If we eliminate the co-efficients b_n etc. after dividing the typical equation by θ_1-4n^2 to secure convergence, we obtain the infinite determinant

$$\begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \frac{-\theta_1}{4^2-\theta_1} & \frac{(i\mu+4)^2}{4^2-\theta_1} & \frac{-\theta_1}{4^2-\theta_1} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \frac{-\theta_1}{2^2-\theta_1} & \frac{(i\mu+2)^2}{2^2-\theta_1} & \frac{-\theta_1}{2^2-\theta_1} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \frac{-\theta_1}{0^2-\theta_1} & \frac{(i\mu)^2}{0^2-\theta_1} & \frac{-\theta_1}{0^2-\theta_1} & 0 & & \cdots \\
\cdots & 0 & 0 & 0 & \frac{-\theta_1}{2^2-\theta_1} & \frac{(i\mu-2)^2}{2^2-\theta_1} & \frac{-\theta_1}{2^2-\theta_1} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \frac{-\theta_1}{4^2-\theta_1} & \frac{(i\mu-4)^2}{4^2-\theta_1} & \frac{-\theta_1}{4^2-\theta_1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}=0$$

$$\text{or } \sum \left[4n(n+1)-r(r+1)\right] A_r P_r e^{2i\sigma_n t}$$

$$+\frac{2C^2}{a}\frac{\partial P_n}{\partial z} n(n+1)P_n e^{2i\sigma_n t} - \frac{C^2}{a}\frac{\partial P_n}{\partial z}\left[2z\frac{\partial P_n}{\partial z}\right.$$

$$\left. - n(n+1) P_n\right] e^{2i\sigma_n t} = 0,$$

$$\text{or } \sum \left[4n(n+1)-r(r+1)\right] A_r P_r e^{2i\sigma_n t}$$

$$= \frac{C^2}{a}\left[2z\left(\frac{\partial P_n}{\partial z}\right)^2 - 3n(n+1)\frac{\partial P_n}{\partial z}\cdot P_n\right] e^{2i\sigma_n t}$$

we shall now expand $2z\left(\frac{\partial P_n}{\partial z}\right)^2 - 3n(n+1)P_n\frac{\partial P_n}{\partial z}$ in a series of zonal harmonics.

Let $f(z) = 2z\left(\frac{\partial P_n}{\partial z}\right)^2 - 3n(n+1)P_n\frac{\partial P_n}{\partial z}$

$$= \frac{\partial P_n}{\partial z}\left[2P_1\{(2n-1)P_{n-1}+(2n-5)P_{n-3}+\ldots\} - 3n(n+1)P_n\right]$$

Now

$$P_1 P_{n-1} = \frac{n}{2n-1} P_n + \frac{n-1}{2n-1} P_{n-2}$$

$$P_1 P_{n-3} = \frac{n-2}{2n-5} P_{n-2} + \frac{n-3}{2n-5} P_{n-4}$$

$$P_1 P_{n-5} = \frac{n-4}{2n-9} P_{n-4} + \frac{n-5}{2n-9} P_{n-6}$$

etc.

$$\therefore f(z) = \frac{\partial P_n}{\partial z}\left[\{2n-3n(n+1)\}P_n + 2(n-3)P_{n-2} + 2(2n-7)P_{n-4} + \ldots\right]$$

$$= [B_{n-1} P_{n-1} + B_{n-3} P_{n-3} + \ldots][D_n P_n + D_{n-2} P_{n-2} + \ldots]$$

$$= C_{2n-1} P_{2n-1} + C_{2n-3} P_{2n-3} + \ldots + C_1 P_1$$

(for all values of n even or odd, the last terms of $f(z)$ will contain P_1 as can be easily seen). Here

$$D_n = -3n^2 - n,$$

$$D_{n-2} = 2(2n-3),$$

$$D_{n-4} = 2(2n-7),$$

etc., and

$$B_{n-1} = 2n-1,$$

$$B_{n-3} = 2n-5,$$

$$B_{n-5} = 2n-9,$$

etc.

Now using Adam's result for the expansion of the product of any two Legendre's co-efficients in terms of the Lengendre's co-efficients, we have

$$C_{2n-1}=D_n B_{n-1}\frac{1\cdot3\cdot5\ldots(2n-3)}{(n-1)\,!}\cdot\frac{(n+1)(n+2)\ldots(2n-1)}{(2n+1)(2n+3)\ldots(4n-1)}\times(4n-1)$$

$$C_{2n-3}=D_n B_{n-1}\frac{1.3.5\ldots(2n-5)}{(n-2)\,!}\cdot\frac{n(n+1)\ldots(2n-2)}{(2n-1)(2n+1)\ldots(4n-3)}\times(4n-5)$$

$$+D_n B_{n-3}\frac{1.3.5\ldots(2n-5)}{(n-3)\,!}\cdot\frac{(n+1)(n+2)\ldots(2n-3)}{(2n+1)(2n+3)\ldots(4n-5)}\times(4n-5)$$

$$+B_{n-1}D_{n-2}\frac{1.3\ldots(2n-5)}{(n-2)\,!}\cdot\frac{n(n+1)\ldots(2n-3)}{(2n-1)(2n+1)\ldots(4n-5)}\times(4n-5).$$

$C_{2n-5}=$etc.

Thus $A_1=\dfrac{c^2}{a}\dfrac{C_1}{[4n(n+1)-1.2]}$,

$$A_3=\frac{c^2}{a}\frac{C_3}{[4n(n+1)-3.4]},$$

$$A_{2n-(2j-1)}=\frac{c^2}{a}\frac{C_{2n-(2j-1)}}{[4n(n+1)-(2n-2j+2)(2n-2j+1)]}$$

The nature of the solution obtained above shows that if ξ is calculated to 2nd order of approximation we get tides of the 2nd order being proportional to c^2 and the frequency being double that of the primary disturbance. Continuing the approximation we obtain tides of higher orders of frequencies 3, 4...times that of the primary.

11. The following particular cases are interesting:—

1. When $n=1$,

$$\xi=CP_1(z)e^{i\sigma_1 t}-\frac{2}{3}\frac{c^2}{a}\cdot P_1(z)e^{2i\sigma_1 t}.$$

and σ_1 is given by $\sigma_1^2=1.2.\dfrac{gh_0}{a^2}$.

2. When $n=2$,

$$\xi=CP_2(z)e^{i\sigma_2 t}-\frac{c^2}{a}\left[\frac{27}{55}\cdot P_1(z)+\frac{21}{10}\cdot P_3(z)\right]e^{2i\sigma_2 t}.$$

and σ_2 is given by $\sigma_2^2=2.3.\dfrac{gh_0}{a^2}$.

3. When $n=3$,

$$\xi=CP_3(z)e^{i\sigma_3 t}-\frac{c^2}{a}\left[\frac{82}{161}P_1(z)+\frac{197}{126}P_3(z)+\frac{250}{63}P_5(z)\right]\times e^{2i\sigma_3 t}.$$

and σ_3 is given by $\sigma_3^2=3.4.\dfrac{gh_0}{a^2}$.

or

$$\Delta(i\mu)=0 \text{ say.}$$

Thus we get $\sin^2\left(\frac{1}{2}\pi i\mu\right)=\Delta(0)\sin^2\left(\frac{1}{2}\pi\sqrt{\theta_1}\right)$.

[See Whittaker's analysis p. 409].

This determines μ. Then b_n is given by $b_n=\frac{L_n}{L_0}b_0$ where L_n =co-factor of b_n in the determinant $\Delta\cdot(i\mu)$ and L_0 =co-factor of b_0.

Thus if the canal communicates with an open sea in which tidal waves

$$\eta=C\cos(\sigma t+\epsilon)$$

are maintained, we have

$$\eta=\frac{Ce^{\mu z}\sum_{n=-\infty}^{n=\infty}L_n e^{2niz}\cos(\sigma t+\epsilon)}{e^{\mu z_1}\sum_{n=-\infty}^{n=\infty}L_n e^{2niz_1}}$$

where z_1 is given by $\sin 2z_1=\frac{c_1}{a}$. c_1 being the distance of the mouth of the canal from the origin.

Part II.

Second and Higher Order Tides.

9. Adopting the Lagrangian plan of making the co-ordinates refer to the individual particles of the fluid, the following equations can be easily established :—

$$\frac{\partial^2\xi}{\partial t^2}=g.\frac{1}{1+\frac{\partial\xi}{\partial x}}\cdot\frac{\partial}{\partial x}\left[\frac{S}{b}\frac{\frac{\partial\xi}{\partial x}}{\left(1+\frac{\partial\xi}{\partial x}\right)}\right], \qquad \dots (1)$$

$$\frac{\eta}{h}=-\frac{\frac{\partial\xi}{\partial x}}{1+\frac{\partial\xi}{\partial x}}, \qquad \dots (2)$$

where the symbols used have the same meaning as before.

10. Let $b=b_0$ and $S=b_0h=b_0h_0\left(1-\frac{x^2}{a^2}\right)$, so that the longitudinal section is a parabola given by $h=h_0\left(1-\frac{x^2}{a^2}\right)$.

From (1), putting $\frac{x}{a}=z$ and neglecting the third order-terms we get

$$\frac{a^2}{gh_0}\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]-\frac{2}{a}\frac{\partial\xi}{\partial z}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]$$

$$-\frac{1}{a}\frac{\partial\xi}{\partial z}\cdot\frac{\partial^2\xi}{\partial z^2} \qquad \text{......(3)}$$

Neglecting square terms in (3) we have

$$\frac{a^2}{gh_0}\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right],$$

Assuming that $\xi\propto e^{i\sigma_n t}$ and putting $\sigma_n{}^2=n(n+1)\frac{gh_0}{a^2}$ we have

$$\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]+n(n+1)\xi=0.$$

This has the solution $\xi=CP_n(z)$, where n is determined by the condition that ξ is finite when $z=\pm1$ whence we see that n is integral.

For any integral value of n, σ_n is given by $\sigma_n{}^2=n(n+1)\frac{gh_0}{a^2}$.

Now reverting to equation (3), assume

$$\xi=CP_n(z)e^{i\sigma_n t}+\Sigma A_rP_r(z)e^{2i\sigma_n t},$$

where A_r is supposed small so that its square can be neglected.

Substituting in (3) and always neglecting $A_r{}^2$ etc., we have

$$-4n(n+1)\Sigma A_rP_re^{2i\sigma_n t}=-\Sigma A_rr(r+1)P_re^{2i\sigma_n t}$$

$$-\frac{2C^2}{a}\frac{\partial P_n}{\partial z}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial P_n}{\partial z}\right]e^{2i\sigma_n t}$$

$$-\frac{C^2}{a}\frac{\partial P_n}{\partial z}(1-z^2)\frac{\partial^2P_n}{\partial z^2}e^{2i\sigma_n t}$$

(2) *The problem of the sphere.*

Consider the three equations

$$\nabla^2 X_x + \frac{1}{1+\sigma}\,\frac{\partial^2 \odot}{\partial x^2} = 0,$$

$$\nabla^2 X_y + \frac{1}{1+\sigma}\,\frac{\partial^2 \odot}{\partial x \partial y} = 0,$$

$$\nabla^2 X_z + \frac{1}{1+\sigma}\,\frac{\partial^2 \odot}{\partial x \partial z} = 0.$$

Multiplying by x, y, z, and adding we have

$$[x\nabla^2 X_x + y\nabla^2 X_y + z\nabla^2 X_z$$

$$+\frac{1}{1+\sigma}\left[x\frac{\partial^2 \odot}{\partial x^2} + y\frac{\partial^2 \odot}{\partial x \partial y} + z\frac{\partial^2 \odot}{\partial x \partial z}\right] = 0,$$

or,

$$\nabla^2 (xX_x + yX_y + zX_z) + \frac{1}{1+\sigma}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{\partial \odot}{\partial x}$$

$$-2\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) = 0.$$

Now since

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$

we have

$$\nabla^2 (rX_r) + \frac{1}{1+\sigma}\,\frac{\partial}{\partial x}\left(r\frac{\partial \odot}{\partial r} - \odot\right) = 0.$$

Similarly we have

$$\left.\begin{aligned} \nabla^2 (rY_r) + \frac{1}{1+\sigma}\,\frac{\partial}{\partial y}\left(r\frac{\partial \odot}{\partial r} - \odot\right) &= 0, \\ \nabla^2 (rZ_r) + \frac{1}{1+\sigma}\,\frac{\partial}{\partial z}\left(r\frac{\partial \odot}{\partial r} - \odot\right) &= 0, \end{aligned}\right\} \quad \dots \quad \dots \quad (2)$$

where $r\dfrac{\partial \odot}{\partial r} - \odot$ is also a harmonic function.

The form of the equations is exactly similar to the preceding equations.

(3) The sphere of radius a, has given tractions X_r, Y_r, Z_r over the surface.

Assuming $\odot = \sum_n \odot_n$,

where $\odot_n$ is a solid homogeneous harmonic of the nth degree,

$$r\frac{\partial \odot}{\partial r} - \odot = \sum (n-1)\odot_n$$

also remembering that

$\nabla^2(r^2-a^2)F_n = 2(2n+3)F_n$, where F_n is a solid homogeneous harmonic of the nth degree, we see that the solutions of the equation (2) can be expressed in the form

$$rX_r = -\frac{1}{2(1+\sigma)}\sum\frac{(n-1)}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial x} + aX_{ro},$$

$$rY_r = -\frac{1}{2(1+\sigma)}\sum\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial y} + aY_{ro},$$

$$rZ_r = -\frac{1}{2(1+\sigma)}\sum\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial z} + aZ_{ro},$$

where X_{ro}, Y_{ro}, Z_{ro}, are harmonic functions which have given values over the surface of the sphere $r=a$, and hence are completely determined. If

$$X_{ro} = \Sigma X_n,\ Y_{ro} = \Sigma Y_n,\ Z_{ro} = \Sigma Z_n\ ;$$

we have

$$rX_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial x} + a\Sigma X_n,$$

$$rY_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial y} + a\Sigma Y_n$$

$$rZ_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial z} + a\Sigma Z_n.$$

The Stress-Equations of Equilibrium

BY

SATYENDRANATH BASU.

[*Read April 6th, 1919.*]

It was shown by Mitchell that the six stress co-efficients in an isotropic medium satisfy six equations of the type :—

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x^2}=0, \qquad \nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y \partial z}=0,$$

... ...

... ...

These equations however have not been used for solving the general problems of Elasticity. It is shown here, that the equations can be successfully integrated, in the case of a semi-infinite body bounded by a plane. In the case of the sphere the equations can be conveniently transformed, in a different form, which then admit of integration in an infinite series of spherical harmonics.

(1) *The semi-infinite solid bounded by* $z=0$.

The surface tractions X_z, Y_z, Z_z; are supposed to have given values over the plane $z=0$.

Consider the equations

$$\nabla^2 X_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x \partial z}=0, \qquad \nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial z^2}=0,$$

$$\nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y \partial z}=0.$$

Since $\odot$ is a harmonic function the general solution can be written as

$$\left.\begin{aligned} X_z &= -\frac{1}{2(1+\sigma)}\, z\,\frac{\partial \odot}{\partial x}+X_{z0}, \\ Y_z &= -\frac{1}{2(1+\sigma)}\, z\,\frac{\partial \odot}{\partial y}+Y_{z0}, \\ Z_z &= -\frac{1}{2(1+\sigma)}\, z\,\frac{\partial \odot}{\partial z}+Z_{z0}, \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (1)$$

where X_{z0}, Y_{z0}, Z_{z0}, are harmonic functions which have given values X_z, Y_z, Z_z, over the plane $z=0$.

The functions are therefore uniquely determined; they are in fact:—

$$X_{z0}=\frac{1}{2\pi}\,\frac{\partial}{\partial z}\iint \frac{X_z}{r}\,dx\,dy,\quad Y_{z0}=\frac{1}{2\pi}\,\frac{\partial}{\partial z}\iint \frac{Y_z}{r}\,dx\,dy,$$

$$Z_{z0}=\frac{1}{2\pi}\,\frac{\partial}{\partial z}\iint \frac{Z_z}{r}\,dx\,dy,$$

also since

$$\frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z}=0,$$

we have from (1)

$$-\frac{1}{2(1+\sigma)}\,\frac{\partial \odot}{\partial z}+\frac{1}{2\pi}\,\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right)=0,$$

where

$$L=\iint \frac{X_z}{r}\,dx\,dy,\quad M=\iint \frac{Y_z}{r}\,dx\,dy,\quad N=\iint \frac{Z_z}{r}\,dx\,dy,$$

and

$$\odot=\frac{1+\sigma}{\pi}\left[\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right].$$

Thus X_z, Y_z, Z_z, $\odot$, are all determined.

The solution may be afterwards completed, and U, V, W found out as in Cerrutti's method.

Again

$$\frac{\partial}{\partial x}(rX_r)+\frac{\partial}{\partial y}(rY_r)+\frac{\partial}{\partial z}(rZ_r)$$

$$=x\left[\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]+y\left[\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right]$$

$$+z\left[\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right]+X_x+Y_y+Z_z=\odot$$

it follows that

$$\odot=-\frac{1}{(1+\sigma)}\Sigma\frac{n-1}{(2n+1)}\left[x\frac{\partial\odot_n}{\partial x}+y\frac{\partial\odot_n}{\partial y}+z\frac{\partial\odot_n}{\partial z}\right]$$

$$+a\Sigma\left(\frac{\partial X_n}{\partial r}+\frac{\partial Y_n}{\partial x}+\frac{\partial Z_n}{\partial z}\right)$$

or

$$\odot=-\frac{1}{1+\sigma}\Sigma\frac{n(n-1)}{2n+1}\odot_n+a\Sigma\left(\frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial y}+\frac{\partial Z_n}{\partial z}\right).$$

So that

$$\Sigma\odot_n\left[\frac{(1+\sigma)(2n+1)+n(n-1)}{(2n+1)(1+\sigma)}\right]=a\Sigma\Psi_n,$$

where

$$\Psi_n=\frac{\partial X_{n+1}}{\partial x}+\frac{\partial Y_{n+1}}{\partial y}+\frac{\partial Z_{n+1}}{\partial z}.$$

So that

$$\odot_n=\frac{a(2n+1)(1+\sigma)}{(1+\sigma)(2n+1)+n(n-1)}\cdot\Psi_n.$$

Thus X_r, Y_r, Z_r, $\odot$, are all determined in terms of the known value of X_r, Y_r, Z_r on the surface.

Review

THEORY AND APPLICATIONS OF FINITE GROUPS—BY PROFS. G. A. MILLER, H. F. BLICHFELDT AND L. E. DICKSON. *New York, Wiley, 1916. 8vo. Pp. xvii+390. Price 17s. net.*

We thankfully acknowledge presentation of a copy of the above work by Prof. G. A. Miller for review in the Bulletin of the Calcutta Mathematical Society. As the book has been in the hands of the mathematical public for some time and has been widely reviewed in Mathematical and Scientific Periodicals already we do not think it would be useful to repeat a general summary of its contents here. The introduction to the book gives an unusually clear insight into the up-to-date features of the work and explains the way in which the work has been divided between the three distinguished authors.

Professor G. A. Miller has done much to advance and popularize the study of group theory among English-knowing students. His efforts in this connection are comparable with those of his distinguished compatriot Professor G. B. Halsted in a different direction. But for Professor G. B. Halsted it is doubtful if the study of non-Euclidean geometry among English-knowing people would have attained the degree of progress and popularity it has acquired to-day. The present work by Professor G. A. Miller and his two learned colleagues has therefore been not quite unexpected.

The English student of the general theory of finite groups has had to depend mainly on two text-books, namely those by Professors Burnside and Hilton. Professor Burnside's classical work is hardly suitable for the beginner in the study of this very difficult subject. Professor Hilton's work is more suitable to the beginner and is almost indispensable to him at present. It contains an admirable series of examples after each article which serve the double purpose of testing as well as leading the learner. It suffers however from over-condensation and is more of the nature of a compendium of facts and formulæ than a lucid exposition of the subject. The present work by three eminent experts has therefore met a necessity.

The proper exposition of a growing body of doctrines, highly abstract in their nature, which have engrafted themselves on almost

every stock of concrete mathematical knowledge is by no means a light task. To attempt to place everything even in a general way into the limited compass of a single readable text-book is to court over-condensation and obscurity. The present authors have wisely limited themselves to a number of simple illustrations of general principles, to the treatment of a number of classic theorems from up-to-date points of view, to the applications to a number of classic problems in a simple and elegant way and to the inclusion of one or two interesting results recently arrived at on their side of the Atlantic. Yet nowhere any special knowledge of mathematics has been presupposed. Professor Miller has followed a simple and elegant style, which is natural to him and which has been well maintained by his two learned colleagues. In the present work the student of the theory of groups will find an interesting companion to which he may revert profitably on occasions when the older and heavier text-books confuse or bore him. It seems to be in the contemplation of the authors to judge from the newly published Finite Collineative Groups by Professor Blichfeldt to issue a number of text-books on special topics of the theory of groups with single authorship in order to supplement this introductory text-book of joint authorship.

The work has been aptly dedicated to C. Jordan from whose fundamental investigations and lucid expositions the authors claim to have drawn inspiration. It is a worthy outcome of the *Traite des Substitutions*.

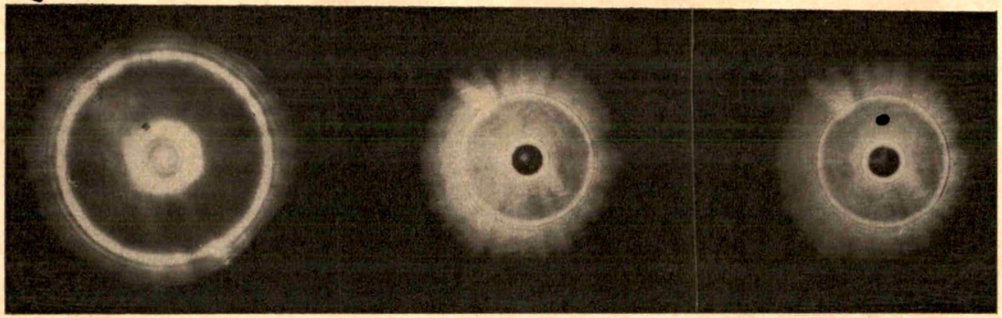
S. M.

Rt. Hon. Lord Rayleigh, O.M., F.R.S.

The Members of the Calcutta Mathematical Society received with deep regret the news of the death of the Rt. Hon. Lord Rayleigh, who was an Honorary Member of the Society since its foundation and took considerable interest in its work. They feel equally with the rest of the scientific world the severe loss which science has sustained by the death of this leader of scientific thought for the last forty years. To several Indian workers, his death would have come as a personal loss, for not a few of them have received personal encouragement from him. His writings have proved and will long continue to be a souice of inspiration to many. It is perhaps fitting here to recall the fact that this world-renowned scientist visited India some years ago and inspected some of our educational institutions.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY, VOL X, No 3, December, 1919.

Fig. 1 Fig. 2 Fig. 3



Illustrating the formation of optical images by a diffracting boundary.

On a special square matrix of order six

BY

C. E. CULLIS

[*Read March 3rd, 1918*]

Summary.—The matrix ψ defined in equation (2) is one which occurs very frequently. Its rank and its reciprocal are determined in Art. 1. The remaining articles contain illustrations of its use. In Arts 4 and 5 the stress-strain relations of an isotropic body are determined without assuming the existence of a strain-energy function, the general argument being a simplification of that contained in Chapter VI of *Love's Elasticity*.

1. *Homogeneous linear transformations of the variables in the matrix*

$$[x^2, y^2, z^2, yz, zx, xy].$$

If we use the notations

$$\phi=[abc]_{123},\ \Delta=\det\phi=(abc)_{123}, \qquad \ldots \quad (1)$$

$$\psi=[e]_6^6=\begin{bmatrix} a_1^2, & b_1^2, & c_1^2, & 2b_1c_1, & 2c_1a_1, & 2a_1b_1 \\ a_2^2, & b_2^2, & c_2^2, & 2b_2c_2, & 2c_2a_2, & 2a_2b_2 \\ a_3^2, & b_3^2, & c_3^2, & 2b_3c_3, & 2c_3a_3, & 2a_3b_3 \\ a_2a_3, & b_2b_3, & c_2c_3, & b_2c_2+b_3c_3, & c_2a_3+c_3a_2, & a_2b_3+a_3b_2 \\ a_3a_1, & b_3b_1, & c_3c_1, & b_3c_1+b_1c_3, & c_3a_1+c_1a_3, & a_3b_1+a_1b_3 \\ a_1a_2, & b_1b_2, & c_1c_2, & b_1c_2+b_2c_1, & c_1a_2+c_2a_1, & a_1b_2+a_2b_1 \end{bmatrix} \quad \ldots \quad (2)$$

and apply to the variables x, y, z the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad \ldots \quad (A)$$

we obtain

$$[x^2, y^2, z^2, yz, zx, xy]=[X^2, Y^2, Z^2, YZ, ZX, XY][e]_6^6. \qquad \ldots \quad (3)$$

When $c_1=\lambda a_1+\mu b_1$, $c_2=\lambda a_2+\mu b_2$, $c_3=\lambda a_3+\mu b_3$, the minor of ψ formed with its 1st, 5th and 6th vertical rows becomes degenerate, and consequently all simple minor determinants of that minor matrix are divisible by Δ. In fact the [illegible]ected minor determinants of ψ of order 3 belonging to that minor, when arranged according to the scheme

(123, 156, 146, 145, 256, 246, 245, 356, 346, 345,

456, 234, 235, 236, 134, 135, 136, 124, 125, 126),

form the matrix

$$\Delta.[-4a_1a_2a_3,\ a_1^3, -a_1^2a_2, -a_1^2a_3,\ a_1a_2^2, -a_2^3,\ a_2^2a_3, -a_1a_3^2, -a_2a_2^3,\ a_3^3\ a_1a_2a_3,\ 0, -2a_2a_3^2,\ 2a_2^2a_3,\ 2a_1a_3^2, 0,\ 2a_1^2a_3,\ 2a_1a_2^2, -2a_1^2a_2,\ 0], \quad \ldots (4)$$

and the co-factors in ψ of these respective determinants form the matrix

$$\Delta.\ [-A_1A_2A_3,\ A_1^3, -A_1^2A_2, -A_1^2A_3,\ A_1A_2^2, -A_2^3,\ A_2^2A_3, -A_1A_3^2, -A_2A_3^2,\ A_3^3,\ 2A_1A_2A_3,\ 0, -A_2A_3^2,\ A_2^2A_3,\ A_1A_3^2,\ 0,\ A_1^2A_3,\ A_1A_2^2, -A_1^2A_2,\ 0], \quad \ldots (4')$$

where $[ABC]_{123}$ is the reciprocal of $[abc]_{123}$. Hence by expanding the determinant $[e]_6^6$ in terms of the minor determinants of order 3 belonging to the 1st, 5th and 6th vertical rows, we see that

$$[e]_6^6 \doteq \det \psi = \Delta^4. \quad \ldots (5)$$

It follows that ψ is undegenerate or degenerate according as ϕ is undegenerate or degenerate.

It is easily seen in a similar way that when ϕ is degenerate, every vertical minor of ψ formed with four vertical rows is degenerate, and therefore the rank of ψ cannot exceed 3. It then follows from (4′) that when ϕ has rank 2, the matrix ψ has rank 3. It is moreover obvious that when ϕ has rank 1 or 0, the matrix ψ has the same rank. Therefore the rank of ψ depends on the rank of ϕ in the way shown in the following scheme.

Rank of ϕ	3	2	1	0
Rank of ψ	6	3	1	0

Again if $[ABC]_{123}$ is the reciprocal of $[abc]_{123}$, and if we write

$$\Psi=[E]_6^6$$

$$=\begin{bmatrix} A_1^2, & B_1^2, & C_1^2, & B_1C_1, & C_1A_1, & A_1B_1 \\ A_2^2, & B_2^2, & C_2^2, & B_2C_2, & C_2A_2, & A_2B_2 \\ A_3^2, & B_3^2, & C_3^2, & B_3C_3, & C_3A_3, & A_3B_3 \\ 2A_2A_3, & 2B_2B_3, & 2C_2C_3, & B_2C_3+B_3C_2, & C_2A_3+C_3A_2, & A_2B_3+A_3B_2 \\ 2A_3A_1, & 2B_3B_1, & 2C_3C_1, & B_3C_1+B_1C_3, & C_3A_1+C_1A_3, & A_3B_1+A_1B_3 \\ 2A_1A_2, & 2B_1B_2, & 2C_1C_2, & B_1C_2+B_2C_1, & C_1A_2+C_2A_1, & A_1B_2+A_2B_1 \end{bmatrix}, \quad (2')$$

we see by direct multiplication that

$$[e]_6^6\, \overline{\underline{E}}_6^6=\overline{\underline{E}}_6^6\, [e]_6^6=\Delta^2[1]_6^6\,; \qquad \ldots \quad (6)$$

and this shows that

the reciprocal of $[e]_6^6$ *is* $\Delta^2\,[E]_6^6$.

When $\Delta\neq 0$, we deduce from (A) the inverse transformation

$$\Delta\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}=\begin{bmatrix} A_1, A_2, A_3 \\ B_1, B_2, B_3 \\ C_1, C_2, C_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \ldots \quad (A')$$

from which it follows that

$$\Delta^2.\,[X^2, Y^2, Z^2, YZ, ZX, XY]=[x^2, y^2, z^2, yz, zx, xy].\,[E]_6^6, \quad (3')$$

as could have been deduced from (3) by prefixing $[E]_6^6$ on both sides.

NOTE.—When ϕ is undegenerate and

$$\begin{bmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{bmatrix} \text{ is the } \textit{inverse} \text{ of } \begin{bmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{bmatrix},$$

the matrix Ψ defined by (2′) is the inverse of the matrix ψ defined by (2), and the two mutually inverse transformations

$$\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} a_1, & b_1, & c_1\\ a_2, & b_2, & c_2\\ a_3, & b_3, & c_3\end{bmatrix}\begin{bmatrix} X\\ Y\\ Z\end{bmatrix}, \quad \begin{bmatrix} X\\ Y\\ Z\end{bmatrix} = \begin{bmatrix} A_1, & A_2, & A_3\\ B_1, & B_2, & B_3\\ C_1, & C_1, & C_3\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix}$$

lead to the two mutually inverse transformations

$$[x^2, y^2, z^2, yz, zx, xy] = [X^2, Y^2; Z^2, YZ, ZX, XY].\ \left\lceil e \right\rfloor^6_6,$$

$$[X^2, Y^2, Z^2, YZ, ZX, XY,] = [x^2, y^2, z^2, yz, zx, xy].\ [E]^6_6.$$

2. *Rank of the matrix*

$$M_5 = f(x, y, z) = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1 z_1, & z_1 x_1, & x_1 y_1\\ x_2^2, & y_2^2, & z_2^2, & y_2 z_2, & z_2 x_2, & x_2 y_2\\ x_3^2, & y_3^2, & z_3^2, & y_3 z_3, & z_3 x_3, & x_3 y_3\\ x_4^2, & y_4^2, & z_4^2, & y_4 z_4, & z_4 x_4, & x_4 y_4\\ x_5^2, & y_5^2, & z_5^2, & y_5 z_5, & z_5 x_5, & x_5 y_5 \end{bmatrix},$$

where (x_1, y_1, z_1), (x_2, y_2, z_2), ..(x_5, y_5, z_5) *are the (projective) co-ordinates of five distinct points* $P_1, P_2, ..P_5$ *in homogeneous 2-way space.*

If a point P has co-ordinates (x, y, z) with respect to one triangle of reference, and co-ordinates (X, Y, Z) with respect to any other triangle of reference, there exists a transformation of the form (A) in which $\Delta \neq 0$, and we have

$$f(x, y, z) = f(X, Y, Z)\ \left\lceil e \right\rfloor^6_6,$$

where $\left\lceil e \right\rfloor^6_6$ is undegenerate. We conclude that the rank of M_5 is the same whatever the triangle of reference may be.

If all the five points $P_1, P_2, \ldots P_5$ are collinear, we can take the straight line on which they lie to be the side $x=0$ of the triangle of reference, and P_1 and P_2 to be corners of the triangle of reference, and suppose the co-ordinates of $P_1, P_2, \ldots P_5$ to be $(0, b, 0)$,

$(0, 0, c)$, $(0, \beta_1, \gamma_1)$, $(0, \beta_2, \gamma_2)$, $(0, \beta_3, \gamma_3)$, when $b, c, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$, are all different from 0. Then M_5 has same rank as

$$\begin{bmatrix} b^2, & 0, & \beta_1^2, & \beta_2^2, & \beta_3^2 \\ 0, & c^2, & \gamma_1^2, & \gamma_2^2, & \gamma_3^2 \\ 0, & 0, & \beta_1\gamma_1, & \beta_2\gamma_2, & \beta_3\gamma_3 \end{bmatrix}.$$

Thus in this case M_5 has rank 3.

If four but not five of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, \beta_1, \gamma_1)$, $(0, \beta_2, \gamma_2)$, where each letter denotes a non-zero quantity. Then M_5 has the same rank as

$$\begin{bmatrix} a^2, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & \beta_1^2, & \beta_2^2 \\ 0, & 0, & c, & \gamma_1^2, & \gamma_2^2 \\ 0, & 0, & 0, & \beta_1\gamma_1, & \beta_2\gamma_2 \end{bmatrix}.$$

Thus in this case M_5 has rank 4.

If three but not four of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, \beta, \gamma)$, (x, y, z), where $a, b, c, \beta, \gamma, x$ are non-zero quantities, and y and z are not both zero. Then M_5 has the same rank as the matrix

$$\begin{bmatrix} a^2, & 0, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & 0, & 0, & 0 \\ 0, & 0, & c^2, & 0, & 0, & 0 \\ 0, & \beta^2, & \gamma^2, & \beta\gamma, & 0, & 0 \\ x^2, & y^2, & z^2, & yz, & zx, & xy \end{bmatrix},$$

which is 4 plus the rank of $[z, y]$. Thus in this case M_5 has rank 5.

If no three of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$,

$(0, b, 0)$, $(0, 0, c)$, $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, where each letter denotes a non-zero quantity. Then M_6 has the same rank as the matrix

$$\begin{bmatrix} a^2, & 0, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & 0, & 0, & 0 \\ 0, & 0, & c^2, & 0, & 0, & 0 \\ \alpha_1^2, & \beta_1^2, & \gamma_1^2, & \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \alpha_2^2, & \beta_2^2, & \gamma_2^2, & \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \end{bmatrix}$$

Since the simple minor determinants of the matrix $\begin{bmatrix} \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \end{bmatrix}$

cannot all vanish, we see that in this case M_6 has rank 5.

Thus in all cases the rank of the matrix $f(x, y, z)$ is given by the following scheme

Maximum number of collinear points	2	3	4	5
Rank of M_6	5	5	4	3

NOTE 1.—For the matrix $\mathsf{M}_4 = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3 \\ x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4 \end{bmatrix}$

we have the following scheme.

Maximum number of collinear points	2	3	4
Rank of M_4	4	4	3

NOTE 2.—The rank of the matrix

$$M_3 = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3 \end{bmatrix}$$

is always 3, when the three points P_1, P_2, P_3 are distinct.

3. *Quadric curve or curves through five given points in homogeneous 2-way space.*

If $P_1, P_2, \ldots P_5$ are five distinct given points whose (projective) co-ordinates are (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots (x_5, y_5, z_5)$, then

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

will be a quadric curve passing through them if and only if the matrix of the co-efficients satisfies the equation

$$M \cdot \begin{bmatrix} a \\ b \\ c \\ f \\ g \\ h \end{bmatrix} = 0, \text{ where } M = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & 2y_1z_1, & 2z_1x_1, & 2x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & 2y_2z_2, & 2z_2x_2, & 2x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & 2y_3z_3, & 2z_3x_3, & 2x_3y_3 \\ x_4^2, & y_4^2, & z_4^2, & 2y_4z_4, & 2z_4x_4, & 2x_4y_4 \\ x_5^2, & y_5^2, & z_5^2, & 2y_5z_5, & 2z_5x_5, & 2x_5y_5 \end{bmatrix}.$$

By article 2 the possible ranks of M are 5, 4 and 3; and therefore the number of unconnected non-zero solutions of the equation is either 1 or 2 or 3. If we disregard the quadric curve of rank 0, we see that:

(1) There is always at least one quadric curve passing though the 5 points.

(2) Except when 4 of the points are collinear, there is only one quadric curve passing through the 5 points.

(3) When 4 of the points are collinear, but not all 5 of them, there are exactly two unconnected quadric curves passing through the 5 points.

(4) When all 5 of the points are collinear, there are three and only three unconnected quadric curves passing through the 5 points.

In case (3) a quadric curve through the 5 points consists of the straight line on which 4 of the points lie and a straight line passing through the other point. In case (4) a quadric curve through the 5 points consists of the straight line on which the 5 points lie and one other straight line.

NOTE.—*Conic or conics through five given coplanar points in common 3-way space.*

The foregoing results are of course true when (x_1, y_1, z_1), (x_2, y_2, z_2), (x_5, y_5, z_5) are the projective co-ordinates of five distinct given coplanar points with respect to a reference framework lying in their plane, the quadric curves being now conics. The points at infinity may as usual be taken to be those for which $z=0$; and some or all the given points may lie on the line at infinity. The 5 given points determine a conic uniquely except when 4 of them are collinear.

4. *The stress-strain relations for an isotropic solid.*

Consider any body, such as a solid, which is slightly strained from a state of zero stress. With the usual notations for the components of strain and stress with reference to rectangular axes (OX, OY, OZ) let

$$[e_1, e_2, e_3, e_4, e_5, e_6]=[e_{xx}, e_{yy}, e_{zz}, \tfrac{1}{2} e_{yz}, \tfrac{1}{2}e_{zx}, \tfrac{1}{2} e_{xy}],$$

$$[\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4, \mathrm{E}_5, \mathrm{E}_6]=[\mathrm{X}_x, \mathrm{Y}_y, \mathrm{Z}_z, \mathrm{Y}_z, \mathrm{Z}_x, \mathrm{X}_y];$$

and let the stress-strain relations for those axes of co-ordinates be

$$\overline{\underline{\mathrm{E}}}_6 = [c]_6^6 \, \overline{\underline{e}}_6. \qquad \ldots \quad (7)$$

Let (OX′, OY′, OZ′) be any other set of rectangular axes through O, the direction-cosines of OX′, OY′, OZ′ with reference to the axes (OX, OY, OZ) being respectively (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3). When this second set of co-ordinate axes is used, let $e_1, e_2, \ldots e_6$, $\mathrm{E}_1, \mathrm{E}_2, \ldots \mathrm{E}_6$ be replaced by $e_1', e_2', \ldots e_6'$, $\mathrm{E}_1', \mathrm{E}_2', \ldots \mathrm{E}_6'$, and let the stress-strain relations be

$$\overline{\underline{\mathrm{E}'}}_6 = [c']_6^6 \, \overline{\underline{e'}}_6 \qquad \ldots \quad (7')$$

Then the body is isotropic (in its unstrained state) if and only if

$$[c']_6^6 = [c]_6^6$$

for all choices of the second set of axes.

Now if

$$[\omega]_6^6 = \begin{bmatrix} l_1^2, & m_1^2, & n_1^2, & 2m_1n_1, & 2n_1l_1, & 2l_1m_1 \\ l_2^2, & m_2^2, & n_2^2, & 2m_2n_2, & 2n_2l_2, & 2l_2m_2 \\ l_3^2, & m_3^2, & n_3^2, & 2m_3n_3, & 2n_3l_3, & 2l_3m_3 \\ l_2l_3, & m_2m_3, & n_2n_3, & m_2n_3+m_3n_2, & n_2l_3+n_3l_2, & l_2m_3+l_3m_2 \\ l_3l_1, & m_3m_1, & n_3n_1, & m_3n_1+m_1n_3, & n_3l_1+n_1l_3, & l_3m_1+l_1m_3 \\ l_1l_2, & m_1m_2, & n_1n_2, & m_1n_2+m_2n_1, & n_1l_2+n_2l_1, & l_1m_2+l_2m_1 \end{bmatrix}; \quad \ldots (8)$$

$$\overline{\underline{\Omega}}{}_6^6 = \begin{bmatrix} l_1^2, & l_2^2, & l_3^2, & 2l_2l_3, & 2l_3l_1, & 2l_1l_2 \\ m_1^2, & m_2^2, & m_3^2, & 2m_2m_3, & 2m_3m_1, & 2m_1m_2 \\ n_1^2, & n_2^2, & n_3^2, & 2n_2n_3, & 2n_3n_1, & 2n_1n_2 \\ m_1n_1, & m_2n_2, & m_3n_3, & m_2n_3+m_3n_2, & m_3n_1+m_1n_3, & m_1n_2+m_2n_1 \\ n_1l_1, & n_2l_2, & n_3l_3, & m_3n_1+m_1n_3, & n_3l_1+n_1l_3, & n_1l_2+n_2l_1 \\ l_1m_1, & l_2m_2, & l_3m_3, & m_1n_2+m_2n_1, & l_3m_1+l_1m_3, & l_1m_2+l_2m_1 \end{bmatrix}, \quad \ldots (8')$$

we have

$$[\omega]_6^6 \, \overline{\underline{\Omega}}{}_6^6 = \overline{\underline{\Omega}}{}_6^6 \, [\omega]_6^6 = [1]_6^6 ,$$

i.e. $[\omega]_6^6$ and $[\Omega]_6^6$ are two mutually inverse undegenerate square matrices; and the formulae for the transformation of strain and stress components are

$$\overline{\underline{e'}}_6 = [\omega]_6^6 \, \overline{\underline{e}}_6 , \quad \overline{\underline{E'}}_6 = [\omega]_6^6 \, \overline{\underline{E}}_6 . \quad \ldots (9)$$

From (7) and (9) it follows that

$$\overline{\underline{E'}}_6 = [\omega]_6^6 \, [c]_6^6 \, \overline{\underline{\Omega}}{}_6^6 \cdot \overline{\underline{e'}}_6 , \quad i.e. \ [c']_6^6 = [\omega]_6^6 \, [c]_6^6 \, \overline{\underline{\Omega}}{}_6^6 .$$

Thus the body is isotropic if and only if we always have

$$[\omega]_6^6 \; [c]_6^6 \; \underline{\overline{\Omega}}_6^6 = [c]_6^6 \qquad \ldots \quad (10)$$

i.e.
$$[\omega]_6^6 [c]_6^6 = [c]_6^6 [\omega]_6^6 \qquad \ldots \quad (10')$$

however the axes (OX′, OY′, OZ′) are chosen.

First let the new axes be formed by merely reversing the axis of x. Then we have

$$[e_1', e_2', e_3', e_4', e_5', e_6'] = [e_1, e_2, e_3, e_4, -e_5, -e_6],$$

$$[E_1', E_2', E_3', E_4', E_5', E_6'] = [E_1, E_2, E_3, E_4, -E_5, -E_6].$$

In this case $[c']_6^6$ is formed by changing the signs of the first four elements in the last two vertical rows and the last two horizontal rows of $[c]_6^6$, and the equation $[c']_6^6 = [c]_6^6$ requires that all those 16 elements shall vanish. From this result and the similar results obtained by reversing the axes of y and z, we see that it is a necessary condition for isotropy that $[c]_6^6$ must have the form

$$[c]_6^6 = \begin{bmatrix} c, & 0 \\ 0, & k \end{bmatrix}_{3,3}^{3,3}, \text{ where } [k]_3^3 = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}. \qquad \ldots \quad (11)$$

If we give it this form, and write

$$[\omega]_6^6 = \begin{bmatrix} u, & p \\ q, & v \end{bmatrix}_{3,3}^{3,3}, \qquad \ldots \quad (12)$$

the necessary and sufficient conditions (10′) can be replaced by

$$[u]_3^3 [c]_3^3 = [c]_3^3 [u]_3^3, \qquad \ldots \quad (13)$$

$$[v]_3^3 [k]_3^3 = [k]_3^3 [v]_3^3, \qquad \ldots \quad (14)$$

$$[p]_3^3[k]_3^3=[c]_3^3[p]_3^3, \qquad \dots (15)$$

$$[q]_3^3[c]_3^3=[k]_3^3[q]_3^3, \qquad \dots (16)$$

where $[u]_3^3$, $[v]_3^3$, $[p]_3^3$, $[q]_3^3$ have the values shown by (8). From (14) it follows at once that another necessary condition is $k_1=k_2=k_3$; and we can therefore write

$$k_1=k_2=k_3=\mu.$$

Next let the new axes (OX′, OY′, OZ′) be formed by turning the axes (OX, OY, OZ) through any angle θ about OZ, so that

$$\begin{bmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_2 \end{bmatrix}=\begin{bmatrix} \cos\theta, & \sin\theta, & 0 \\ -\sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix}$$

Then the necessary equation (13) is

$$\begin{bmatrix} \cos^2\theta, & \sin^2\theta, & 0 \\ \sin^2\theta, & \cos^2\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} \cos^2\theta, & \sin^2\theta, & 0 \\ \sin^2\theta, & \cos^2\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix},$$

and this equation is satisfied for all values of θ if and only if

$$c_{21}=c_{12}, \quad c_{31}=c_{32}, \quad c_{11}=c_{22}.$$

From this result and the corresponding results obtained from rotations about OX, OY, we see that it must be possible to write

$$c_{23}=c_{32}=c_{31}=c_{13}=c_{12}=c_{21}=\lambda,\quad c_{11}=c_{22}=c_{33}=\lambda+\rho,$$

and that the condition (13) is then always satisfied.

Finally the conditions (15) and (16) are satisfied when and only when

$$\rho=\mu.$$

Thus the necessary and sufficient condition for isotropy in the unstrained state is that $[c]_6^6$ shall have the form

$$[c]_6^6=\begin{bmatrix} \lambda+\mu, & \lambda, & \lambda, & 0, & 0, & 0 \\ \lambda, & \lambda+\mu, & \lambda, & 0, & 0, & 0 \\ \lambda, & \lambda, & \lambda+\mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & \mu, & 0, & 0 \\ 0, & 0, & 0, & 0, & \mu, & 0 \\ 0, & 0, & 0, & 0, & 0, & \mu \end{bmatrix}.$$

When we replace μ by 2μ, and write $e_{xx}+e_{yy}+e_{zz}=\Delta$, the corresponding stress-strain relations are

$$X_x=\lambda\Delta+2\mu\, e_{xx},\quad Y_y=\lambda\Delta+2\mu\, e_{yy},\quad Z_z=\lambda\Delta+2\mu\, e_{zz}.$$

$$Y_z=\mu e_{yz},\qquad Z_x=\mu e_{zx},\qquad X_y=\mu e_{xy}.$$

5. *The stress-strain relations for any isotropic body.*

If, as must be the case when the body is not a solid, the unstrained state is not one of zero stress, we must replace (7) by

$$\overline{\begin{matrix} E \\ 1 \end{matrix}}_{6,1}=\begin{bmatrix} c, & d \\ 0, & 1 \end{bmatrix}_{6,1}^{6,1}\overline{\begin{matrix} e \\ 1 \end{matrix}}_{6,1}, \qquad \ldots \quad (18)$$

and (10′) by

$$\begin{bmatrix} \omega, & 0 \\ 0, & 1 \end{bmatrix}_{6,1}^{6,1} \begin{bmatrix} c, & d \\ 0, & 1 \end{bmatrix}_{6,1}^{6,1} = \begin{bmatrix} c, & d \\ 0, & 1 \end{bmatrix}_{6,1}^{6,1} \begin{bmatrix} \omega, & 0 \\ 0, & 1 \end{bmatrix}_{6,1}^{6,1} . \qquad \ldots \quad (19)$$

The conditions that the body shall be isotropic in the unstrained state are the same as before together with the additional condition

$$[\omega]_6^6 \; [d]_6^1 = [d]_6^1 \qquad \ldots \quad (20)$$

Considering the three particular cases in which the direction of one of the three axes of co-ordinates is merely reversed, we see from (20) that we must have

$$d_{41} = d_{51} = d_{61} = 0, \qquad \ldots \quad (21)$$

and these values reduce (20) to

$$[u]_3^3 \; [d]_3^1 = [d]_3^1, \; 0 = [q]_3^3 \; [d]_3^1 \qquad \ldots \quad (22)$$

Considering the cases of rotations about OX, OY, OZ, we see that the first of the conditions (22) can only be satisfied when we can write

$$d_{11} = d_{21} = d_{31} = -p,$$

and then both conditions are always satisfied.

Thus any slightly strained body whatever is isotropic in the unstrained state if and only if the stress-strain relations are

$$\begin{bmatrix} X_x \\ Y_y \\ Z_z \\ Y_z \\ Z_x \\ X_y \end{bmatrix} = \begin{bmatrix} \lambda+2\mu, & \lambda, & \lambda, & 0, & 0, & 0, & -p \\ \lambda, & \lambda+2\mu, & \lambda, & 0, & 0, & 0, & -p \\ \lambda, & \lambda, & \lambda+2\mu, & 0, & 0, & 0, & -p \\ 0, & 0, & 0, & \mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & \mu, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & \mu, & 0 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ 1 \end{bmatrix},$$

or

$$X_x=\lambda\Delta+2\mu\, e_{xx}-p, \quad Y_y=\lambda\Delta+2\mu\, e_{yy}-p, \quad Z_z=\lambda\Delta+2\mu\, e_{zz}-p,$$

$$Y_z=\mu\, e_{yz}, \qquad Z_x=\mu\, e_{zx}, \qquad X_y=\mu\, e_{xy}.$$

On the formation of optical images by a diffracting boundary.

[With a Plate]

BY

BHUPENDRA CHANDRA DAS.

[*Read January 26th, 1919.*]

In a recent paper[1] published in the Philosophical Magazine, Prof. Banerji has noticed that if a circular aperture placed in front of a lens is illuminated by a point source of light and if a small screen is placed in the focal plane so as to cut off the entire geometrical cone of rays, a bright image of the source may be traced along the axis behind the screen and for a considerable distance beyond. As remarked by him, this phenomenon is somewhat analogous to that observed by Porter[2] and Hufford[3] almost simultaneously, namely, that, the rays diffracted by a circular disk can form an optical image of the source along the axis of symmetry, but differs from it, as in this case the image is formed by the rays diffracted by the boundary of a circular aperture. At the suggestion of Prof. Banerji I undertook a detailed study of this phenomenon and have succeeded in obtaining a mathematical theory. While carrying out the experiment, I have observed that the central bright spot is surrounded by two sets of alternately bright and dark rings, the outer set being at a considerable distance apart from the inner one. A large - number of very faint rings may also be observed in the space between the first set of rings and the second, but not between the first set of rings and the bright spot, which region is marked by almost complete darkness. It appears that the configuration of the outer set of rings depends to a considerable extent on the form of the screen placed

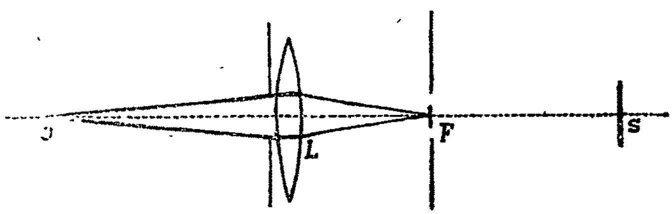
[1] Banerji, "On the radiation of light from the boundaries of diffracting apertures," *Phil. Mag.*, vol. xxxvii., Jan. 1919.

[2] Porter, "On the formation of images by means of an opaque disk," *Phil. Mag.*, vol. xxvii, p. 673 (1914).

[3] Hufford, "Some new diffraction photographs," *Phys Rev.* vol. III, Ser. 2, p 241 (1914).

in the focal plane. These rings undergo modification if the screen in the focal plane be either displaced or replaced by one having a different boundary. Figures I and II in the plate illustrate the phenomenon observed with a circular diffracting aperture illuminated by a strong beam of light allowed to pass through a circular pin hole. The central bright spot has a circular shape in these photographs. The formation of optical images by a circular diffracting boundary illuminated by a non-circular point source of light is illustrated in figure III. In this photograph the source is triangular and the central bright spot will be noticed to have also a triangular shape. In any case the image is found to closely follow the form of the source.

Let r be the radius of the circular aperture of a lens L which is illuminated by a point source of light O and let the beam converge to a focus F at a distance b from the aperture. Let the light at the focal plane be cut off by means of an opaque circular disk of small radius ζ_1. For the sake of giving definiteness to the problem, we shall impose an upper limit to ζ, say ζ_2, that is to say, we shall suppose the diffracted rays to pass through an annular opening in the focal plane.



The disturbance at any point P (ζ, ϕ), in the focal plane is known to be

$$\frac{K J_1\left(\frac{mr}{b}\zeta\right)}{\zeta} \sin nt,$$

where K is a constant and $m=\frac{2\pi}{\lambda}$, λ being the wave length of light.

The disturbance at any point Q, distant x from the axis, on a screen S placed at any distance c behind the focal plane can be regarded as due to the diffracted rays which pass through the annulas.

Now if R be the distance of Q from P, then

$$R^2=(\zeta\cos\phi-x)^2+\zeta^2\sin^2\phi+c^2$$

$$=\zeta^2-2\zeta x\cos\phi+x^2+c^2.$$

The effect at Q due to an elementary disturbance at P is, by Huyghen's principle, equal to

$$\zeta d\zeta\, d\phi\ K\ \frac{J_1\left(\frac{mr}{b}\zeta\right)}{\zeta}\cdot\frac{1}{R}\sin(mR-nt).$$

Hence the total disturbance at Q is given by

$$\Psi=K\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2}\frac{1}{R}\ J_1\left(\frac{mr}{b}\zeta\right)\sin(mR-nt)\ d\zeta d\phi.$$

If the opening be small, and Q is very near the axis we may substitute $\frac{1}{c}$ for $\frac{1}{R}$, and the expression for the total disturbance becomes

$$\Psi=\frac{K}{c}\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2}J_1\left(\frac{mr}{b}\zeta\right)\sin(mR-nt)\ d\zeta d\phi.$$

Since x and ζ are very small campared to b or c, we have, on extracting the square root and neglecting terms of the higher order,

$$R=c+\frac{\zeta^2}{2c}+\frac{x^2}{2c}-\frac{\zeta x}{c}\cos\phi.$$

Putting $m\left(c+\frac{x^2}{2c}\right)-nt=\omega$, we can write

$$\Psi=\frac{K}{c}\left[C\sin\omega+S\cos\omega\right], \qquad \dots \quad \text{(i)}$$

where

$$C=\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2}J_1\left(\frac{mr}{b}\zeta\right)\cos\left(\frac{m\zeta^2}{2c}-\frac{mx}{c}\zeta\cos\phi\right)d\zeta d\phi,$$

$$S=\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2}J_1\left(\frac{mr}{b}\zeta\right)\sin\left(\frac{m\zeta^2}{2c}-\frac{mx}{c}\zeta\cos\phi\right)d\zeta d\phi.$$

On integrating with respect to ϕ we easily get,

$$C=2\pi\int_{\zeta_1}^{\zeta_2} J_1(A\zeta)\,J_0(B\zeta)\cos\left(\frac{1}{2}\mu\zeta^2\right)d\zeta, \qquad \ldots \quad \text{(ii)}$$

$$S=2\pi\int_{\zeta_1}^{\zeta_2} J_1(A\zeta)\,J_0(B\zeta)\sin\left(\frac{1}{2}\mu\zeta^2\right)d\zeta, \qquad \ldots \quad \text{(iii)}$$

where

$$A=\frac{mr}{b},\ B=\mu x,\ \mu=\frac{m}{c}.$$

We now proceed to evaluate these integrals. Integrating by parts, without putting in the limits for the present, we obtain

$$\int J_1(A\zeta)\,J_0(B\zeta)\cos\left(\frac{1}{2}\mu\zeta^2\right)d\zeta=\cos\left(\frac{1}{2}\mu\zeta^2\right)\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta$$

$$+\mu\int\zeta\sin\left(\frac{1}{2}\mu\zeta^2\right)\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta\,d\zeta. \qquad \ldots \quad \text{(iv)}$$

Now remembering the formulae

$$\left.\begin{aligned} &\int x^n J_{n-1}(x)\,dx=x^n J_n(x), \\ \therefore\ &\frac{d}{dx}\left[\frac{J_n(x)}{x^n}\right]=-\frac{J_{n+1}(x)}{x^n}, \end{aligned}\right\} \qquad \ldots \quad \text{(v)}$$

we get

$$\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta=\frac{A}{B^2}\int\frac{J_1(A\zeta)}{A\zeta}.B\zeta\,J_0(B\zeta)\,d(B\zeta)$$

$$=\frac{A}{B^2}\cdot\frac{J_1(A\zeta)}{A\zeta}.B\zeta\,J_1(B\zeta)+\frac{A^2}{B^2}\int\frac{J_2(A\zeta)}{A\zeta}.B\zeta\,J_1(B\zeta)\,d\zeta$$

$$=\frac{1}{A}\cdot\frac{A}{B}J_1(A\zeta)\,J_1(B\zeta)+\frac{A^3}{B^4}\int\frac{J_2(A\zeta)}{(A\zeta)^2}\cdot(B\zeta)^2\,J_1(B\zeta)\,d(B\zeta)$$

and by successive application of (v) this becomes finally

$$= \frac{1}{A}\left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta) + \frac{A^3}{B^3} J_3 (A\zeta) J_3 (B\zeta) + \dots\right].$$

On substitution of the above, the second term of the right hand side of (iv) becomes

$$\frac{\mu}{A}\int \zeta \left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta) + \frac{A^3}{B^3} J_3 (A\zeta) J_3 (B\zeta) + \dots\right] \sin (\tfrac{1}{2}\mu\zeta^2)\, d\zeta$$

$$= \frac{\mu}{A} \sin (\tfrac{1}{2}\mu\zeta^2) \int \zeta \left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta) + \dots\right] d\zeta$$

$$+ \frac{\mu^2}{A}\int \zeta \cos (\tfrac{1}{2}\mu\zeta^2) \int \zeta \left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta) + \dots\right] d\zeta.\, d\zeta.$$

Again, proceeding as in the previous case,

$$\int \zeta \frac{A}{B} J_1 (A\zeta) J_1 (B\zeta)\, d\zeta = \frac{\zeta}{B}\left[\frac{A}{B} J_1 (A\zeta) J_2 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_3 (B\zeta) + \dots\right],$$

$$\int \zeta \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta)\, d\zeta = \frac{\zeta}{B}\left[\frac{A^2}{B^2} J_2 (A\zeta) J_3 (B\zeta) + \dots\right]$$

Hence

$$\int \zeta \left[\frac{A}{B} J_1(A\zeta) J_1(B\zeta) + \frac{A^2}{B^2} J_2(A\zeta) J_2(B\zeta) + \dots \right] d\zeta$$

$$= \frac{\zeta}{B} \left[\frac{A}{B} J_1(A\zeta) J_2(B\zeta) + 2\frac{A^2}{B^2} J_2(A\zeta) J_3(B\zeta) \right.$$

$$\left. + 3\frac{A^3}{B^3} J_3(A\zeta) J_4(B\zeta) + \dots \right]$$

Proceeding to perform the integrations in this way we finally arrive at the result,

$$\int J_1(A\zeta) J_0(B\zeta) \cos(\tfrac{1}{2}\mu\zeta^2)\, d\zeta = \frac{1}{A} \cos(\tfrac{1}{2}\mu\zeta^2) \left[D_0 - \left(\frac{\mu\zeta}{B}\right)^2 D_2 \right.$$

$$\left. + \left(\frac{\mu\zeta}{B}\right)^4 D_4 - \dots \right] + \frac{1}{A} \sin(\tfrac{1}{2}\mu\zeta^2) \left[\left(\frac{\mu\zeta}{B}\right) D_1 - \left(\frac{\mu\zeta}{B}\right)^3 D_3 + \dots \right],$$

where

$$D_0 = \frac{A}{B} J_1(A\zeta) J_1(B\zeta) \div \frac{A^2}{B^2} J_2(A\zeta) J_2(B\zeta)$$

$$+ \frac{A^3}{B^3} J_3(A\zeta) J_3(B\zeta) + \dots$$

$$D_1 = \frac{A}{B} J_1(A\zeta) J_2(B\zeta) + 2\frac{A^2}{B^2} J_2(A\zeta) J_3(B\zeta)$$

$$+ 3\frac{A^3}{B^3} J_3(A\zeta) J_4(B\zeta) + \dots$$

$$D_2 = \frac{A}{B} J_1(A\zeta) J_3(B\zeta) + 3\frac{A^2}{B^2} J_2(A\zeta) J_4(B\zeta)$$

$$\div 6\frac{A^3}{B^3} J_3(A\zeta) J_5(B\zeta) + \dots$$

If we put

$$M = D_0 - \left(\frac{\mu\zeta}{B}\right)^2 D_2 + \left(\frac{\mu\zeta}{B}\right)^4 D_4 - \ldots$$

$$N = \left(\frac{\mu\zeta}{B}\right) D_1 - \left(\frac{\mu\zeta}{B}\right)^3 D_3 + \ldots$$

we can write,

$$C = \frac{2\pi}{A}\left[M \cos(\tfrac{1}{2}\mu\zeta^2) + N \sin(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2}$$

and exactly in the same way,

$$S = \frac{2\pi}{A}\left[M \sin(\tfrac{1}{2}\mu\zeta^2) - N \cos(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2}.$$

The terms here are expressed in ascending powers of $\frac{A}{B}$ and $\frac{\mu}{B}$ and the results therefore hold good only when these are less than unity, in which case the series are all convergent. Near the axis however, where we require our results to hold good, x is very small and so also is B and thus $\frac{A}{B}$ is very large. We shall in these cases have to express our results in a series of $\frac{B}{A}$. It is easy to do this by the previous method, provided we note that in evaluating the integrals like $\int J_1(A\zeta)J_0(B\zeta)d\zeta$ by parts, we shall have to integrate $J_1(A\zeta)$ and differentiate $J_0(B\zeta)$ instead of the reverse method adopted in the previous case. The final result can be written in the form

$$C = \frac{2\pi}{A}\left[-P \cos(\tfrac{1}{2}\mu\zeta^2) - Q \sin(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2},$$

$$S = \frac{2\pi}{A}\left[-P \sin(\tfrac{1}{2}\mu\zeta^2) + Q \cos(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2},$$

where

$$P = E_0 - \left(\frac{\mu\zeta}{A}\right)^2 E_2 + \left(\frac{\mu\zeta}{A}\right)^4 E_4 - \ldots$$

$$Q = \left(\frac{\mu\zeta}{A}\right) E_1 - \left(\frac{\mu\zeta}{A}\right)^3 E_3 + \ldots$$

and

$$E_0 = J_0(A\zeta)J_0(B\zeta) + \frac{B}{A} J_1(A\zeta) J_1(B\zeta) + \frac{B^2}{A^2} J_2(A\zeta) J_2(B\zeta) + \ldots,$$

$$E_1 = J_1(A\zeta) J_0(B\zeta) + 2\frac{B}{A} J_2(A\zeta) J_1(B\zeta) + 3\frac{B^2}{A^2} J_3(A\zeta) J_2(B\zeta) + \ldots,$$

$$E_2 = J_2(A\zeta) J_0(B\zeta) + 3\frac{B}{A} J_3(A\zeta) J_1(B\zeta) + 6\frac{B^2}{A^2} J_4(A\zeta) J_2(B\zeta) + \ldots,$$

etc.

If I denote the intensity of illumination, then

$$I = C^2 + S^2.$$

The only quantity involving x is B which is $= \mu r$, and to discuss the maxima or minima on the axis where $r=0$, we are to remember that,

$$\operatorname*{Lt.}_{x\to 0} J_0(x) = 1, \quad \operatorname*{J_1}_{x\to 0}(x) = \operatorname*{J_2}_{x\to 0}(x) = \ldots = 0.$$

We now easily get,

$$\operatorname*{E_0}_{x\to 0} = J_0(A\zeta), \quad \operatorname*{E_1}_{x\to 0} = J_1(A\zeta), \quad \operatorname*{E_2}_{x\to 0} = J_2(A\zeta), \ldots \text{etc.} \ldots$$

$$\underset{x\to 0}{\frac{\partial E_0}{\partial x}} = \underset{x\to 0}{\frac{\partial E_1}{\partial x}} = \underset{x\to 0}{\frac{\partial E_2}{\partial x}} = \ldots = 0$$

Also since

$$\operatorname*{J_1'}_{x\to 0}(x) = \tfrac{1}{2},$$

we get on a second differentiation and substitution of $x=0$,

$$\operatorname*{Lt.}_{x\to 0} \frac{\partial^2 E_0}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 J_0(A\zeta) = -\tfrac{1}{2}\mu^2\zeta^2 \operatorname*{E_0}_{x\to 0};$$

$$\operatorname*{Lt.}_{x\to 0} \frac{\partial^2 E_1}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \operatorname*{E_1}_{x\to 0},$$

$$\operatorname*{Lt.}_{x\to 0} \frac{\partial^2 E_2}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \operatorname*{E_2}_{x\to 0},$$

... ...

Thus

$$\underset{x\to 0}{\mathrm{Lt.}}\ P = J_0(A\zeta) - \left(\frac{\mu\zeta}{A}\right)^2 J_2(A\zeta) + \left(\frac{\mu\zeta}{A}\right)^4 J_4(A\zeta) - \ldots,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ Q = \left(\frac{\mu\zeta}{A}\right) J_1(A\zeta) - \left(\frac{\mu\zeta}{A}\right)^3 J_3(A\zeta) + \ldots,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial P}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial Q}{\partial x} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial^2 P}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{P}, \quad \underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial^2 Q}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{Q}$$

Denoting the values of

$$\frac{2\pi}{A}\left[-P\cos(\tfrac{1}{2}\mu\zeta^2) - Q\sin(\tfrac{1}{2}\mu\zeta^2)\right]$$

by C_1 and C_2 when ζ_1 and ζ_2 respectively are substituted for ζ, and the corresponding values of

$$\frac{2\pi}{A}\left[-P\sin(\tfrac{1}{2}\mu\zeta^2) + Q\cos(\tfrac{1}{2}\mu\zeta^2)\right]$$

by S_1 and S_2 we get

$$C = C_2 - C_1, \quad S = S_2 - S_1,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial C}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial S}{\partial x} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial^2 C}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta_2^2 C_2 + \tfrac{1}{2}\mu^2\zeta_1^2 C_1$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial^2 S}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta_2^2 S_2 + \tfrac{1}{2}\mu^2\zeta_1^2 S_1$$

Hence

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial I}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\ \frac{1}{2}\left\{C\frac{\partial C}{\partial x} + S\frac{\partial S}{\partial x}\right\} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\ \frac{\partial^2 I}{\partial x^2} = \underset{x\to 0}{\mathrm{Lt.}}\ \frac{1}{2}\left\{C\frac{\partial^2 C}{\partial x^2} + S\frac{\partial^2 S}{\partial x^2} + \left(\frac{\partial C}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2\right\}$$

$$= \underset{x\to 0}{\mathrm{Lt.}}\ -\tfrac{1}{2}\mu^2\Big[(C_2 - C_1)(\zeta_2^2 C_2 - \zeta_1^2 C_1)$$

$$+ (S_2 - S_1)(\zeta_2^2 S_2 - \zeta_1^2 S_1)\Big].$$

The intensity is maximum for all points on the axis provided $\frac{d^2 I}{dx^2}$, when $x=0$, is negative. It will appear from the above expression that this quantity is negative when $\zeta_2 = \zeta_1$ and will continue to remain negative even when ζ_2 and ζ_1 differ from each other. The exact stage at which the expression may change from negative to a positive value will depend on the actual magnitudes of C_2, C_1 and S_2, S_1. This gives a qualitative explanation of the formation of images behind the screen.

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On Joachimsthal's Attraction Problem

BY

SASINDRA CHANDRA DHAR

[Read March 9th, 1919]

1. Given the total attraction at any point due to an infinite homogeneous rod as an arbitrary function of the perpendicular distance of the point, Joachimsthal's problem [1] consists in determining the law of attraction. Assuming that the law of force is $\psi(r)$, Joachimsthal succeeded in solving the problem for the case when $\psi(r)$ is expansible in a series of negative powers of r. In the present paper, I have shown that the problem can be solved without making this assumption. I have also given the solution of an allied problem which involves the determination of the law of density in an infinite rod, when the total attraction at any point is given as an arbitrary function of the perpendicular distance of the point from the rod, the law of force being the Newtonian Law.

2. The problem as given by Joachimsthal [2] is the following :—

The elements of an infinitely long and homogeneous rod AB attracts the point O whose distance from the rod is h, according to an unknown function of r, *i.e.*, $\psi(r)$. To find this function, if by observation, the total attraction along the perpendicular OM is known to be $F(h)$, where $OM=h$.

If, for the sake of simplicity, the mass of the particle O, the density of the rod and the constant of attraction be taken equal to unity, then the attraction along OP due to an element dt at P on the rod $(MP=t)$ is

$$\psi(r) \cdot dt,$$

and its resolved part along OM is

$$\psi(r) \cos\theta \, dt,$$

θ being the angle MOP.

[1] Joachimsthal : *Crelle's Journal*, Bd. 58.

[2] My attention was drawn to this problem by Dr. Ganesh Prasad, to whom I beg to express my obligations.

Joachimsthal . *loc-cit.*

Therefore, the total attraction along OM due to the infinite rod AB is

$$2h\int_h^\infty \frac{\psi(r)\,dr}{\sqrt{r^2-h^2}}\,.$$

If we write $f(h)$ for $F(h)/2h$, we are required to solve the following integral equation of the first kind :—

$$f(h)=\int_h^\infty \frac{\psi(r)\,dr}{\sqrt{r^2-h^2}}\,. \qquad \ldots \quad (1)$$

The solution which Professor Joachimsthal has obtained is based on the assumption that $\psi(r)$, the unknown function, is capable of expansion in negative integral powers of r. It is possible to obtain the solution of the integral equation (1) without making this assumption.

3. Examining the form of the solution of the integral equation (1) as obtained by Joachimsthal, we may assume, (following a method of Goursat),[1] that

$$\psi(r)=\int_r^\infty \frac{r\phi(x)\,dx}{\sqrt{x^2-r^2}}\,, \qquad \ldots \quad (2)$$

where $\phi(x)$ is the unknown expression whose form is to be determined in such a way that the equation (1) is satisfied.

If we put $r=hr'$, we get from (1) and (2)

$$f(h)=\int_1^\infty \frac{\psi(hr')\,dr'}{\sqrt{r'^2-1}}\,,$$

$$\psi(hr')=\int_{hr'}^\infty \frac{hr'\phi(x)dx}{\sqrt{x^2-h^2r'^2}}\,.$$

[1] E. Goursat "Sur un probleme D'inversion Resolue Par Abel," *Acta Mathematica*, Vol. 27, 1903.

Hence in the latter integral, if we put $x=hx'$, we get

$$\psi(hr')=h\int_{x'}^{\infty}\frac{r'\,\phi(hx')\,dx'}{\sqrt{x'^2-r'^2}}.$$

Therefore,

$$f(h)=h\int_{1}^{\infty}\frac{r'dr'}{\sqrt{r'^2-1}}\int_{x'}^{\infty}\frac{\phi(hx')\,dx'}{\sqrt{x'^2-r'^2}},$$

which, on changing the order of integration (by Dirichlet's formula) becomes

$$f(h)=h\int_{1}^{\infty}\phi(hx')\,dx'\int_{1}^{x'}\frac{r'dr'}{\sqrt{(r'^2-1)\,(x'^2-r'^2)}}$$

$$=\frac{h\pi}{2}\int_{1}^{\infty}\phi(hx')\,dx'=\frac{\pi}{2}\int_{h}^{\infty}\phi(x)\,dx$$

whence

$$\phi(x)=\frac{-2}{\pi}f'(x). \qquad \dots \quad (3)$$

Thus the form of $\phi(x)$ is determined and the solution of the integral equation (1) is obtained in the form

$$\psi(r)=\frac{-2}{\pi}\int_{r}^{\infty}\frac{rf'(x)\,dx}{\sqrt{x^2-r^2}}=\frac{-2r}{\pi}\int_{r}^{\infty}\frac{d\left(\frac{F(x)}{x}\right)}{\sqrt{x^2-r^2}}, \qquad \dots \quad (4)$$

It should be noted that the integral equation (1) has a solution, only if $f(h)=0$, when $h=\infty$. When this condition is satisfied, its solution is given by (4).

4. The solution of the integral equation (1) may also be obtained in series, by following the method of Volterra. By a suitable transformation it is possible to write (1) in the form.

$$\psi(z)=-\frac{\sqrt{2z}}{\pi}\phi'(z)+\frac{\sqrt{2z}}{\pi}\int_{z}^{\infty}K_1(z,r)\,\psi(r)\,dr, \qquad \dots \quad (5)$$

where

$$\phi(z)=\int_z^\infty \frac{f(x)\,dx}{\sqrt{z-x}},$$

$$K(z,r)=\int_z^r \frac{dx}{\sqrt{z-x}\sqrt{r^2-x^2}}, \quad (r>z)$$

and

$$K_1(z,r)=\frac{\partial K}{\partial z}.$$

The equation (5) is the well-known form of Voltera's integral equation of the second kind, whose solution in series can be easily obtained.

5. The problem connected with that of Joachimsthal is the following :—

The density of an infinitely long rod AB varies according to a certain law, *viz.*, a function $\psi(t)$ of the distance t from M the foot of the perpendicular from the external point O. To find the function, if by observation, the total attraction of a mass at O, which attracts according to the inverse square of the distance, along the perpendicular OM is known.

We take, for the sake of brevity, the mass at O and the constant of attraction to be unity.

The mass of an element of matter dt at P is $\psi(t)dt$ and hence the attraction along OP due to this mass is $\psi(t)dt/r^2$. Therefore the total attraction along the perpendicular OM is, if we take the density function $\psi(t)$ to be symmetrical about the point M, given by

$$2h\int_0^\infty \frac{\psi(t)\,dt}{(h^2+t^2)^{\frac{3}{2}}}.$$

If the observed total attraction is given by $f(h)$, then we get the following integral equation of the first kind

$$f(h)=2h\int_0^\infty \frac{\psi(t)\,dt}{(h^2+t^2)^{\frac{3}{2}}}, \qquad \ldots (6)$$

where $\psi(t)$ is the unknown function whose form is required.

6 It should be observed from the study of the integral equation (6) that the necessary condition in order that a solution may exist is that $f(h)=0$ when $h=\infty$, which means physically that there should be no total attraction, when the unit mass O is situated at an infinite distance from the rod. This condition is evidently satisfied in this case.

If we suppose $\psi(t)$ to be of the form

$$\psi(t)=\int_{\sqrt{h^2+t^2}}^\infty \frac{\sqrt{h^2+t^2}.\ x\phi(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}, \qquad \ldots (7)$$

where $\phi(x)$ is an unknown function, whose form is to be determined in such a way that the equation (6) is satisfied. Therefore we get

$$f(h)=2h\int_0^\infty \frac{dt}{(h^2+t^2)^{\frac{3}{2}}}\int_{\sqrt{h^2+t^2}}^\infty \frac{\sqrt{h^2+t^2}.\ x\phi(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}$$

$$=2h\int_h^\infty \frac{dt}{y\sqrt{y^2-h^2}}\int_y^\infty \frac{x\phi(x)\,dx}{\sqrt{x^2-y^2}},$$

(where $y^2=h^2+t^2$)

$$=2h\int_h^\infty x\phi(x)\,dx\int_h^\infty \frac{dy}{y\sqrt{y^2-h^2}\sqrt{x^2-y^2}} \qquad \ldots (8)$$

(on changing the order of integration)

Now, the integral

$$\int_h^x \frac{dy}{y\sqrt{y^2-h^2}\sqrt{x^2-y^2}} = \frac{1}{x}\int_{h/x}^1 \frac{dy_1}{y_1\sqrt{(x^2y_1^2-h^2)(1-y_1^2)}}$$

$(y=xy_1)$.

$$=\frac{1}{x^2}\int_{h_1}^1 \frac{dy_1}{y_1\sqrt{y_1^2-h_1^2)(1-y_1^2)}},$$

(when $h=xh_1$)

$$=\frac{-1}{2x^2}\int_{1/h_1^2}^1 \frac{d\xi}{\sqrt{(\xi-1)(1-h_1^2\xi)}} \quad \left(\text{where } \xi=\frac{1}{y_1^2}\right)$$

$$=\frac{-1}{2hx}\int_{\frac{1}{h_1^2}}^1 \frac{d\xi}{\sqrt{(\xi-1)\left(\frac{1}{h_1^2}-\xi\right)}}$$

If we put

$$\xi=\frac{1}{h_1^2}+\left(1-\frac{1}{h_1^2}\right)\omega,$$

the above integral becomes equal to

$$\frac{1}{2hx}\int_0^1 \frac{d\omega}{\sqrt{\omega}\sqrt{1-\omega}}=\frac{\pi}{2hx}. \qquad \ldots \quad (9)$$

Hence from (8) we get

$$\therefore \quad f(h)=\pi\int_h^\infty \phi(x)\,dx.$$

Therefore

$$\phi(x)=\frac{-1}{\pi}f'(x).$$

Therefore, the solution of the integral equation (6) is given by

$$\psi(t)=-\frac{\sqrt{h^2+t^2}}{\pi}\int_{\sqrt{h^2+t^2}}^\infty \frac{xf'(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}, \qquad \ldots \quad (10)$$

which gives the unknown law of density.

On the potentials of heterogeneous incomplete Ellipsoids and Elliptic Discs.

BY

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1. The potentials of heterogeneous ellipsoids have engaged the attention of many eminent mathematicians and solutions have been obtained by them in definite integrals and in series of spherical harmonics. These are the two most important forms in which the potential function for an ellipsoid has hitherto been expressed. The density in most cases is taken to be of the form

$$\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^{\lambda-1} f(x, y, z)$$

and the law of attraction is assumed to be the inverse nth power of the distance. While the case of the complete ellipsoid has thus been studied quite at length by many writers, the potential of an incomplete ellipsoid has not as yet received proper attention. The late Prof. Grube while giving a solution for the potential of a complete ellipsoid indicated a method by which the potential of a uniform incomplete ellipsoid may be determined.[1] The object of this paper is to study the potentials of heterogeneous incomplete ellipsoids and elliptic discs and to notice incidentally the potentials of ellipsoids with certain discontinuous distributions of mass.

The results given here have all been obtained by the use of discontinuous factors, a method first applied by Dirichlet in the determination of the potential of a uniform ellipsoid. The principle is this.[2] The potential is given by the integral

$$V=\iiint\frac{\sigma dv}{r}$$

[1] Crelle, Vols. 69 and 98.

[2] Kronecker, Vorlesungen über bestimmte Integrale.

taken all throughout the volume of the ellipsoid. This integral has been multiplied by Dirichlet by

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\theta.\cos s\theta}{\theta}\,d\theta$$

which has the value unity if $s<1$ and zero if $s>1$. If $s=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$ then for all points inside the ellipsoid $s=1$, the integral is unity and for all points outside it is zero.

Hence after multiplication we can take the previous integral throughout the entire infinite space and

$$V=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma\frac{\sin\theta\cos s\theta\,d\theta}{\theta}\;\frac{dx\,dy\,dz}{r}.$$

This quadruple integral can now be reduced to a single integral.

Later on Dr. Hobson[1] by using a different integral suggested by Kronecker[2] obtained by the application of the same method the potential of a heterogeneous ellipsoid of n dimensions for a more general law of attraction (inverse $(m+1)$th power of the distance).

In the present paper use has been made of Dr. Hobson's integral which is multiplied by a second discontinuous factor chosen according to the condition of the problem and by the application of the method followed by Kronecker and Hobson the potential function is ultimately obtained as a surface integral.

2. We propose to take up the case of the semi-ellipsoid first as it furnishes an important example of the method under consideration. The discontinuous factors to be used now are the following :—

$$(i)\quad \int_{-\infty}^{\infty} \frac{e^{c(q+iw)}}{(q+iw)^\lambda}\,dw^3 = \frac{2\pi}{\Gamma(\lambda)}c^{\lambda-1} \quad \text{if } c>0$$

$$\text{or}\quad 0 \quad \text{if } c<0 \qquad \lambda>1 \ (\text{q positive})$$

[1] *Proc. Lond. Math. Soc.* vol. 27.

[2] Vorlsungen, *l.c.*

[3] This integral has been used by Hobson, *vide ante*.

$$(ii)\ \frac{1}{\pi}\int_0^\infty \frac{\sin \eta v}{v}\,dv+\frac{1}{2} \quad = \quad 1 \quad \text{if } \eta>0$$

$$\text{or} \quad 0 \quad \text{if } \eta<0.$$

Suppose that the semi-ellipsoid (a, b, c) lies on the positive side of the plane $\eta=0$ and is of density

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1} F\left(\frac{\xi}{a}, \frac{\eta}{b}, \frac{\zeta}{c}\right) \qquad (\lambda-1 \geq 0)$$

$$=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1} e^{\alpha\xi+\beta\eta+\gamma\zeta}\, F_o\left(\frac{x'}{a}, \frac{y'}{b}, \frac{z'}{c}\right)$$

where α, β, γ stand for $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}$ respectively and F_o means that after the differentiations are performed x', y', z' are all put equal to zero.

Now consider the integral

$$P=\frac{1}{m}\ \frac{\Gamma(\lambda)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)^\lambda}$$

$$\times\left[\frac{1}{\pi}\int_0^\infty\frac{\sin \eta v}{v}\,dv+\tfrac{1}{2}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{\alpha\xi+\beta\eta+\gamma\zeta}}{r^m}\, d\xi d\eta d\zeta dw F_o,$$

where

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

It is the potential of a semi-ellipsoid (for which $\eta>0$) of density σ under a law of force varying as the inverse $(m+1)$th power of the distance at the point x, y, z. This we write in the form

$$P=\tfrac{1}{2}P_1+V,$$

where P_1 is the potential of the complete ellipsoid of density σ and

$$V=\frac{1}{m}\ \frac{\Gamma(\lambda)}{2\pi^2}\iiiint\!\!\int\frac{\sin \eta v}{v}\,\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)^\lambda}$$

$$\frac{e^{\alpha\xi+\beta\eta+\gamma\zeta}}{r^m}\, d\xi d\eta d\zeta dv dw\ F_o.$$

It easily follows from the integral (*ii*) which may also be written in the form

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \eta v}{v}\, dv = 1 \quad \text{if } \eta>0$$
$$\text{or } -1 \quad \text{if } \eta<0,$$

that 2V is the potential of an ellipsoid with a discontinuous distribution of mass on opposite sides of the plane $\eta=0$, *viz.*, of the distribution of a volume density σ throughout the half-ellipsoid for which $\eta>0$ and of $-\sigma$ throughout the other half given by $\eta<0$. As the form of σ has been assumed to be such that the potential of the complete ellipsoid for this distribution is known[1] we shall confine our investigation to the determination of the function V only.

3. Now

$$\frac{1}{r^m} = \frac{1}{\Gamma\left(\frac{m}{2}\right)}\int_0^\infty t^{\frac{m}{2}-1} e^{-tr^2}\, dt$$

Hence

$$V = \frac{1}{2\pi^2 m}\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)}\iiiint\!\!\iint \frac{\sin \eta v}{v}\frac{e^{c(q+iw)}}{(q+iw)^\lambda}$$

$$e^{\alpha\xi+\beta\eta+\gamma\zeta}\, e^{-t[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]}$$

$$t^{\frac{m}{2}-1}\, d\xi\, d\eta\, d\zeta\, dv\, dw\, dt\, F_o,$$

at this stage we integrate with respect to ξ, η, ζ respectively and we have the three following integrals :

$$\int_{-\infty}^{\infty} e^{-\left(t+\frac{q+iw}{a^2}\right)\xi^2+2\left(\frac{\alpha}{2}+tx\right)\xi}\, d\xi$$

$$= \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{a^2}}}\, e^{\left(tx+\frac{\alpha}{2}\right)^2\Big/\left(t+\frac{q+iw}{a^2}\right)}$$

[1] Hobson, *l.c.* Dyson. *Quarterly Journal* 1891 vol. xxv.

$$\int_{-\infty}^{\infty} e^{-\left(t+\frac{q+iw}{c^2}\right)\zeta^2+2\left(\frac{\gamma}{2}+tz\right)\zeta}\, d\zeta = \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{c^2}}}\, e^{\left(tz+\frac{\gamma}{2}\right)^2 \Big/ \left(t+\frac{q+iw}{c^2}\right)}$$

and

$$\int_{-\infty}^{\infty} e^{-\left(t+\frac{q+iw}{b^2}\right)\eta^2+2\left(\frac{\beta}{2}+ty\right)\eta} \sin \eta v \, d\eta = \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{b^2}}}$$

$$e^{\left[\left(ty+\frac{\beta}{2}\right)^2-\frac{1}{4}v^2\right] \Big/ \left(t+\frac{q+iw}{b^2}\right)} \sin\left[\frac{\left(ty+\frac{\beta}{2}\right)v}{t+\frac{q+iw}{b^2}}\right].$$

Hence

$$V=\frac{\Gamma(\lambda)}{2\pi^2 m\Gamma\left(\frac{m}{2}\right)}\iiint \frac{1}{v}\,\frac{e^{(q+iw)}}{(q+iw)^\lambda}\, t^{\frac{m}{2}-1}\, e^{-t(x^2+y^2+z^2)}$$

$$\times\frac{\pi\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{a^2}}\ \sqrt{t+\frac{q+iw}{b^2}}\ \sqrt{t+\frac{q+iw}{c^2}}}$$

$$\times e^{\frac{\left(tx+\frac{\alpha}{2}\right)^2}{t+\frac{q+iw}{a^2}}+\frac{\left(ty+\frac{\beta}{2}\right)^2}{t+\frac{q+iw}{b^2}}+\frac{\left(tz+\frac{\gamma}{2}\right)^2}{t+\frac{q+iw}{c^2}}}$$

$$\times e^{-\frac{v^2}{4}\Big/\left(t+\frac{q+iw}{t^2}\right)} \sin\frac{\left(ty+\frac{\beta}{2}\right)v}{t+\frac{q+iw}{b^2}}\, dv\, dw\, dt\, F_o\,;$$

now changing the variable from t to θ by the substitution

$$\frac{q+iw}{t}=\theta,$$

we have

$$V=\frac{abc}{2\sqrt{\pi m}}\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{e^{(q+iw)\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)}}{(q+iw)^{\lambda-\frac{m}{2}+\frac{3}{2}}\,\theta^{\frac{m}{2}-\frac{1}{2}}}$$

$$e^{\frac{a^2\alpha x}{a^2+\theta}+\frac{b^2\beta y}{b^2+\theta}+\frac{c^2\gamma z}{c^2+\theta}}\ \frac{1}{v}\ \frac{e^{\frac{1}{4(q+iw)}\left(\frac{a^2\alpha^2\theta}{a^2+\theta}+\frac{b^2\beta^2\theta}{b^2+\theta}+\frac{c^2\gamma^2\theta}{c^2+\theta}\right)}}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$e^{-\frac{v^2}{4}\frac{\theta b^2}{(q+iw)(\theta+b^2)}}\sin v\left[\frac{b^2y}{b^2+\theta}+\frac{1}{2}\frac{\beta b^2\theta}{(q+iw)(b^2+\theta)}\right]dw\,d\theta\,dv\,F_o.$$

(1) First suppose $\alpha=\beta=\gamma=0$, *i.e.*, $\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1}$;

then

$$V=\frac{1}{m}\frac{abc}{2\sqrt{\pi}}\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)}\int\int\frac{\theta^{\frac{1}{2}-\frac{m}{2}}}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]}}{(q+iw)^{\lambda-\frac{m}{2}+\frac{3}{2}}}\,dw\,d\theta$$

$$\times\int_0^{\infty}e^{-\frac{v^2}{4}\frac{b^2\theta}{(q+iw)(b^2+\theta)}}\frac{\sin\frac{vb^2y}{b^2+\theta}}{v}\,dv.$$

Putting $\quad\frac{b^2y}{b^2+\theta}v=u,\ \frac{dv}{v}=\frac{du}{u}$

and since

$$\int_0^{\infty}e^{-\alpha^2v^2}\cos\beta v\,dv=\frac{\sqrt{\pi}}{2\alpha}e^{-\frac{\beta^2}{4\alpha^2}},$$

$$\int_0^{\infty}\int_0^{\beta}e^{-\alpha^2v^2}\cos\beta v\,d\beta\,dv=\frac{\sqrt{\pi}}{2\alpha}\int_0^{\beta}e^{-\frac{\beta^2}{4\alpha^2}}\,d\beta.$$

i.e.
$$\int_0^\infty e^{-a^2v^2}\frac{\sin\beta v}{v}\,dv=\frac{\sqrt{\pi}}{2a}\int_0^\infty e^{-\frac{\beta^2}{4a^2}}\,d\beta,$$

and
$$\int_0^\infty e^{-a^2v^2}\frac{\sin v}{v}\,dv=\frac{\sqrt{\pi}}{2a}\int_0^1 e^{-\frac{s^2}{4a^2}}\,ds,$$

we have
$$\int_0^\infty e^{-\frac{v^2}{4}\frac{b^2\theta}{(\mathrm{q}+iw)\,(b^2+\theta)}}\sin\frac{b^2y}{b^2+\theta}\,v\,dv$$
$$=\sqrt{\pi}\,\frac{by}{\sqrt{\theta(b^2+\theta)}}(\mathrm{q}+iw)^{\frac{1}{2}}\int_0^1 e^{-\frac{b^2y^2}{\theta(b^2+\theta)}(\mathrm{q}+iw)s^2}\,ds.$$

Hence
$$\mathrm{V}=\frac{1}{m}\,\frac{\Gamma(\lambda)}{2\Gamma\left(\frac{m}{2}\right)}\,ab^2c\,y\int_0^1\int_0^\infty\frac{\theta^{-\frac{m}{2}}\,d\theta\,ds}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$
$$\int_{-\infty}^{\infty}\frac{e^{(\mathrm{q}+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]}}{(\mathrm{q}+iw)^{\lambda-\frac{m}{2}+1}}\,dw.$$

But
$$\int_{-\infty}^{\infty}\frac{e^{(\mathrm{q}+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]}}{(\mathrm{q}+iw)^{\lambda-\frac{m}{2}+1}}\,dw$$
$$=\frac{2\pi}{\Gamma\left(\lambda-\frac{m}{2}+1\right)}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}},$$
$$\left(\lambda-\frac{m}{2}+1>0\right),$$

or 0 according as
$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)},$$

is positive or negative.

Hence

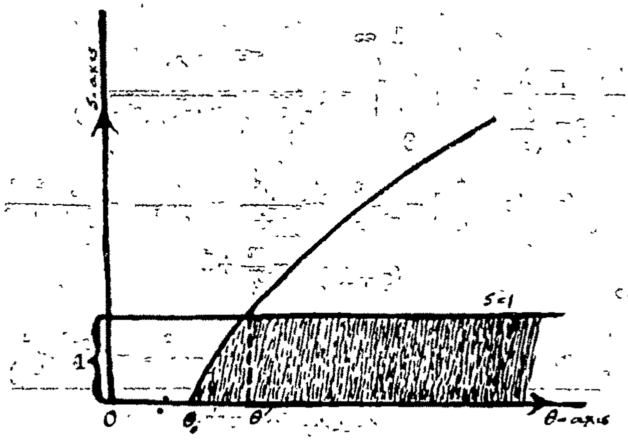
$$V=\frac{1}{m}\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)}\pi ab^2cy\iint\frac{\theta^{-\frac{m}{2}}}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}}ds d\theta,$$

and the integration is to be taken for those values of s and θ which make

$$R=1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0,\ (1\geqq s\geqq 0,\ \theta \text{ positive}).$$

4. The above inequality defines the limits to which the integration should be confined. The potential function in this case is expressed as a surface integral and the region of integration will be evident from the following considerations :



(i) When the point (x,y,z) is outside the ellipsoid it is evident on writing the inequality in the form

$$R\equiv\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)-\frac{s^2b^2}{\theta(b^2+\theta)}y^2>0$$

that the expression within the bracket in R will be negative so long as θ is less then θ_0 which is the parameter of the confocal ellipsoid passing through the point (x,y,z), *i.e.*, the greatest root of the equation

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}=0.$$

So that no real value of s will make R zero or positive. From the value θ_o onwards upto infinity the expression within bracket will be positive (its maximum value being 1), hence we can chose s such that R may be positive and for any definite value of θ the maximum value of s that will make R positive is the value of s that makes R vanish. Now it is also obvious from the equation $R=0$ that s which vanishes with $\theta=\theta_o$, will henceforth inerease continuously and will approach infinity with θ. If we now draw the line $s=1$ the area of integration is easily found. The point where the curve cuts the line $s=1$ is given by the equation

$$1-\frac{x^2}{a^2+\theta'}-\frac{y^2}{b^2+\theta'}-\frac{z^2}{c^2+\theta'}-\frac{b^2y^2}{\theta'(b^2+\theta')}=0.$$

Hence we can express V as the sum of two such integrals as

$$\int_{\theta_o}^{\theta'}\int_{s=0}^{\frac{\sqrt{\theta(b^2+\theta)}}{by}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)^{\frac{1}{2}}} \mathrm{I}\,ds d\theta+\int_{\theta'}^{\infty}\int_{s=0}^{1} \mathrm{I}\,ds d\theta$$

where the integrand is denoted by I.

(*ii*) When the point (x,y,z) is inside the ellipsoid the expression within the bracket will be positive for all values of θ from zero upto infinity and hence we can always find s such that R vanishes. The origin lies on the curve $R=0$ and as θ gradually increases s also continuously increases along with it and ultimately goes to infinity with θ. The region of integration can now be easily obtained by drawing the line $s=1$ and the function V is expressible as the sum of the two integrals

$$\int_{0}^{\theta'}\int_{0}^{\frac{\sqrt{\theta(b^2+\theta)}}{by}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)^{\frac{1}{2}}} \mathrm{I}\,ds d\theta+\int_{\theta'}^{\infty}\int_{0}^{1} \mathrm{I}\,ds d\theta.$$

5. The case of the semi-ellipsoid of uniform density under Newtoniam law of attraction is important and may be deduced from

the result given in §3 by putting $\lambda=1$ and $m=1$.

$$V=2ab^2cy\iint\frac{d\theta ds}{(b^2+\theta)\sqrt{\theta(a^2+\theta)}\,(c^2+\theta)}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\frac{1}{2}}$$

for the limits defined as before by $R>0$.

This may be reduced to a single integral by integrating according to the scheme of the previous article. For

$$\frac{by}{\sqrt{\theta(b^2+\theta)}}\int\frac{ds}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{(b^2+\theta)}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\frac{1}{2}}=\frac{1}{\Delta}\int\beta\sqrt{a^2-\beta^2s^2}\,ds$$

where $$\Delta=\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}\ ,\ \beta=\frac{by}{\sqrt{\theta(b^2+\theta)}}$$

and $$a^2=1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta};$$

this integral as equal to

$$\frac{\beta}{\Delta}\left[\frac{s\sqrt{\frac{a^2}{\beta^2}-s^2}}{2}+\tfrac{1}{2}\,\frac{a^2}{\beta^2}\,\sin^{-1}\frac{\beta s}{a}\right];$$

so that

$$\therefore\int_0^{\frac{a}{\beta}}\beta\sqrt{a^2-\beta^2s^2}\,ds=\frac{\pi}{4}\cdot\frac{a^2}{\beta}.$$

and $$\int_0^1\beta\sqrt{a^2-\beta^2s^2}\,ds=\tfrac{1}{2}\sqrt{a^2-\beta^2}+\tfrac{1}{2}\frac{a^2}{\beta}\sin^{-1}\frac{\beta}{a}.$$

Hence

$$\therefore V=\tfrac{1}{2}\pi abc\int_{0,\,\theta_0}^{\theta'}\frac{d\theta}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)$$

$$+ab^2cy\int_{\theta'}^{\infty}\frac{d\theta}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}\Bigg[\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{b^2y^2}{\theta(b^2+\theta)}\right)^{\frac{1}{2}}$$

$$+\frac{\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)\sqrt{\theta(b^2+\theta)}}{by}\sin^{-1}\frac{by}{\sqrt{\theta(b^2+\theta)}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)^{\frac{1}{2}}}\Bigg].$$

The lower limit of the first integral being 0 or θ_o according as the point is inside or outside the ellipsoid, θ_o and θ' being defined as in §4. It should be remembered that the potential of the semi ellipsoid

$$P=\tfrac{1}{2}\text{ potential of the complete ellipsoid}+V.$$

6. Now, going back to the integral in §3 let us suppose that a, β, γ do not vanish. We have to deal with the integral

$$\int_0^{\infty} e^{-\frac{v^2}{4}\cdot\frac{b^2\theta}{(b^2+\theta)\,(q+iw)}}\,\frac{\sin v\left\{\frac{b^2y}{b^2+\theta}+\frac{1}{2}\frac{\beta b^2\theta}{(b^2+\theta)\,(q+iw)}\right\}}{v}\,dv.$$

This by the substitution

$$\left\{\frac{b^2y}{b^2+\theta}+\tfrac{1}{2}\frac{\beta b^2\theta}{(b^2+\theta)\,(q+iw)}\right\}v=w$$

transforms into

$$\sqrt{\pi}\,\frac{b^2y\,(q+iw)+\frac{1}{2}\beta b^2\theta}{b\sqrt{\theta(b^2+\theta)}\,\sqrt{q+iw}}\int_0^1 \epsilon^{-\frac{[b^2y(q+iw)+\frac{1}{2}\beta b^2\theta]^2}{b^2(b^2+\theta)\theta(q+iw)}s^2}\,ds.$$

Hence

$$V=\frac{\Gamma(\lambda)}{2m\Gamma\left(\frac{m}{2}\right)}\,abc\int_0^{\infty}\int_0^1\frac{\theta^{\frac{1}{2}-\frac{m}{2}}}{(b^2+\theta)\sqrt{\theta(a^2+\theta)\,(c^2+\theta)}}$$

$$e^{a\frac{a^2x}{a^2+\theta}+\beta\frac{b^2y\,(1-s^2)}{b^2+\theta}+\gamma\frac{c^2z}{c^2+\theta}}\,Q\,d\theta ds\;F,$$

where

$$Q=\int_{-\infty}^{\infty} [b^2y(q+iw)+\tfrac{1}{2}\beta b\theta]$$

$$\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{b^2s^2y^2}{\theta(b^2+\theta)}\right]-\frac{1}{4(q+wi)}\left[\frac{a^2\theta}{a^2+\theta}\alpha^2+\frac{b^2\theta}{c^2+\theta}(1-s^2)\beta^2+\frac{c^2\theta}{c^2+\theta}\gamma^2\right]}}{(q+iw)^{\lambda-\frac{m}{2}+2}}\, dw$$

$$=\int_{-\infty}^{\infty}\left[\frac{by}{(q+iw)^{\lambda-\frac{m}{2}+1}}+\frac{1}{2}\frac{\beta b\theta}{(q+iw)^{\lambda-\frac{m}{2}+2}}\right]$$

$$e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\text{etc...}\right]}\left[1+\sum_{1}^{\infty}\frac{1}{n\,!}\frac{A^n}{(q+iw)^n}\right]dw,$$

where

$$A=\frac{\theta}{4}\left[\frac{a^2}{a^2+\theta}\alpha^2+\frac{b^2\beta^2}{b^2+\theta}(1-s^2)+\frac{c^2\gamma^2}{c^2+\theta}\right]=Q_1+Q_2,$$

where
$$Q_1=by\int e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\text{etc.}\right]}\left\{\frac{1}{(q+iw)^{\lambda-\frac{m}{2}+1}}+\sum_{1}^{\infty}\frac{1}{n\,!}\frac{A^n}{(q+iw)^{\lambda-\frac{m}{2}+n+1}}\right\}dw$$

$$=by\,\frac{2\pi}{\Gamma(\lambda-\frac{m}{2}+1)}R^{\lambda-\frac{m}{2}}\left[1-\frac{RB\theta}{2(2\lambda-m+2)}+\frac{R^2B^2\theta^2}{2\cdot4\cdot(2\lambda-m+2)(2\lambda-m+4)}+\dots\right]\dots\dots(\lambda-\tfrac{m}{2}+1>0)$$

where R is as defined in §3 and

$$B=\frac{4A}{\theta}=\frac{a^2}{a^2+\theta}\alpha^2+\frac{b^2\beta^2}{b^2+\theta}(1-s^2)+\frac{c^2\gamma^2}{c^2+\theta}\ \text{ and also}$$

when R is positive, for negative values of R the integral vanishes, and in a similar manner

$$Q_2=\tfrac{1}{2}\beta\, b\theta\ \frac{2\pi}{\Gamma(\lambda-\frac{m}{2}+2)}\,R^{\lambda-\frac{m}{2}+1}\left[1+\frac{RB\theta}{2.\,(2\lambda-m+4)}+\dots\right]\dots$$

where R is positive, and is zero when R is negative.

Now replacing α, β, γ by $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}$ and allowing the whole integral to operate on $F\left(\frac{x'}{a}, \frac{y'}{b}, \frac{z'}{c}\right)$ and finally putting x', y', z' equal to zero after differentiation we have

$$V=V_1+V_2$$

where

$$V_1=\frac{1}{m}\frac{\Gamma(\lambda)\,\pi ab^2c}{\Gamma(\frac{m}{2})\Gamma(\lambda-\frac{m}{2}+1)}\,y\int\!\!\int\frac{\theta^{\frac{1}{2}-\frac{m}{2}}}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}$$

$$\times R^{\lambda-\frac{m}{2}}\left[1+\frac{RL\theta}{2.\,(2\lambda-m+4)}+\dots\right]$$

$$F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2+\theta}(1-s^2), \frac{cz}{c^2+\theta}\right)ds\,d\theta$$

and $$V_2=\frac{1}{2m}\frac{\Gamma(\lambda)}{\Gamma(\frac{m}{2})}\cdot\frac{\pi ac}{\Gamma(\lambda-\frac{m}{2}+2)}\int\!\!\int\frac{\theta^{1-\frac{m}{2}}}{(1-s^2)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R^{\lambda-\frac{m}{2}+1}\left[1+\frac{RL\theta}{2.\,(2\lambda-m+4)}+\dots\right]$$

$$\frac{\partial}{\partial y}F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2+\theta}(1-s^2), \frac{cz}{c^2+\theta}\right)ds\,d\theta$$

where

$$L=\frac{a^2+\theta}{a^2}\frac{\partial^2}{\partial x^2}+\frac{b^2+\theta}{b^2(1-s^2)}\frac{\partial^2}{\partial y^2}+\frac{c^2+\theta}{c^2}\frac{\partial^2}{\partial z^2},$$

and as before the integrations are to be extended over those values of θ and s which make

$$R=1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0\quad(1\geq s\geq 0,\ \theta\text{ positive}).$$

7. The case of the semi-elliptic plate comes in this connection and can be treated exactly in the same way. By using the same discontinuous factors and after a similar series of processes the following results may be obtained.

For a semi-elliptic plate of density σ where

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1} \text{ on the half } \eta>0,$$

the potential $P=\frac{1}{2}P_1+V$, where P_1 is the potential of the entire elliptic plate of this density and

$$V=\frac{1}{m}\frac{\Gamma(\lambda)}{\Gamma(\frac{m}{2})\Gamma(\lambda-\frac{m}{2}+\frac{1}{2})}\sqrt{\pi}ab^2\, y\iint\frac{\theta^{-\frac{m}{2}-\frac{1}{2}}}{(\theta+b^2)(\theta+a^2)^{\frac{1}{2}}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}-\frac{1}{2}}d\theta ds$$

where the integrations are to be taken for all values of s and θ which make

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0. \quad [1\geqq s\geqq 0,\ \theta \text{ positive}].$$

The discussion about the limits of integration given in §4 would also apply in this case. Supposing the plate to be homogeneous, the potential under Newtonian law of force is given by

$$V=ab^2y\iint\frac{d\theta ds}{\theta(b^2+\theta)(a^2+\theta)^{\frac{1}{2}}}, \text{ the limits being properly taken}$$

$$=ab\int_{0,\theta_0}^{\theta'}\frac{d\theta}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}\right]^{\frac{1}{2}}$$

$$+ab^2y\int_{\theta'}^{\infty}\frac{d\theta}{\theta(b^2+\theta)(a^2+\theta)^{\frac{1}{2}}}.$$

according to the scheme of integration given in § 4, and the lower limit of the first integration being 0 or θ_0 according as the point is inside or outside the plate and θ_0 and θ' are the greatest (positive) roots of the equations

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}=0$$

and

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}\left(1+\frac{b^2}{\theta}\right)-\frac{z^2}{\theta}=0$$

respectively.

The case of the density

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1} \mathrm{F}\left(\frac{\xi}{a}, \frac{\eta}{b}\right)$$

can be treated exactly as the corresponding three dimensional problem and results analogous to those given in §6 can be obtained.

8. Having done with the semi-ellipsoid we propose to take up the case of an incomplete ellipsoid bounded by a plane parallel to one of the principal planes. By slightly altering the second discontinuous factor given in §2 the potential of the incomplete figure may be obtained by an exactly similar analysis. Supposing that the solid under contemplation is greater than a semi-ellipsoid and is limited on the upper part by the plane $\eta=\kappa$ the suitable discontinuous factor would be

$$\frac{1}{\pi}\int_0^\infty \frac{\sin(\kappa-\eta)v}{v}\,dv$$

which is $\frac{1}{2}$ or $-\frac{1}{2}$ according as $\eta<\kappa$ or $>\kappa$.

If the density is given by the function

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1} \mathrm{F}\left(\frac{\xi}{a}, \frac{\eta}{b}, \frac{\zeta}{c}\right)$$

the potential P of the figure is given by

$$\mathrm{P}=\tfrac{1}{2}\mathrm{P}_1+\mathrm{V}$$

where P_1 is the potential of the complete ellipsoid of the same density and

$$\mathrm{V}=\frac{1}{m}\frac{\Gamma(\lambda)}{2\pi^2}\int\!\!\int\!\!\int\!\!\int\!\!\int \frac{\sin(\kappa-\eta)v}{v}.\ \frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(\mathrm{q}+iw)}}{(\mathrm{q}+iw)^{\lambda}}.$$

$$\frac{e^{\alpha\xi+\beta\eta+\gamma\zeta}}{r^m}\,d\xi d\eta d\zeta dv dw\ \mathrm{F}.$$

and it may also be shown as in § 6 that

$$\mathrm{V}=\mathrm{V}_1+\mathrm{V}_2.$$

where

$$V_1 = \frac{1}{m} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)} \cdot \pi ac \iint \frac{[k(b^2+\theta)-b^2y]\theta^{-\frac{m}{2}}}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R'^{\lambda-\frac{m}{2}}\left[1+\frac{R'L'\theta}{2\cdot(2\lambda-m+2)}+\dots\dots\dots\right]$$

$$F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2+\theta}(1-s^2)+\frac{ks^2}{b}, \frac{cz}{c^2+\theta}\right) ds\, d\theta$$

and

$$V_2 = -\frac{1}{2m} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+2\right)} \pi ac \iint \frac{\theta^{1-\frac{m}{2}}}{(1-s^2)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R'^{\lambda-\frac{m}{2}+1}\left[1+\frac{R'L'\theta}{2(2\lambda-m+4)}+\dots\dots\dots\right]$$

$$\frac{\partial}{\partial y} F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2-\theta}(1-s^2)+\frac{ks^2}{b}, \frac{cz}{c^2+\theta}\right) ds\, d\theta$$

where

$$L' = \frac{a^2+\theta}{a^2}\frac{\partial^2}{\partial x^2} + \frac{b^2+\theta}{b^2(1-s^2)}\frac{\partial^2}{\partial y^2} + \frac{c^2+\theta}{c^2}\frac{\partial^2}{\partial z^2}$$

and the limits of integration are to be determined from

$$R' \equiv 1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{[k(b^2+\theta)-b^2y]^2}{b^2\theta(b^2+\theta)}s^2>0, \quad \begin{pmatrix} 1\geq s\geq 0 \\ \theta \text{ positive}\end{pmatrix}.$$

A scheme of integration analogous to that given in §4 may be worked out in this case also. The two dimensional problem may be dealt with in a similar manner.

The potential of a complete homogeneous ellipsoid may be deduced from the results given above. This will also serve as a test of the

accuracy of our results. Putting $\lambda=1$, $m=1$ and $F=1$ the potential under Newtonian law of attraction.

$$P = \tfrac{1}{2} P_1 + 2ac \iint \frac{[k(b^2+\theta)-b^2y]}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{\{k(b^2+\theta)-b^2y\}^2}{b^2\theta(b^2+\theta)}s^2\right]^{\frac{1}{2}} ds\, d\theta$$

for the limits defined by $R' > 0$, and when the ellipsoid is complete $k=b$. Following the scheme of integration given in §4 with only a slight modification necessary on this occasion it is evident that in the present case θ' will tend to infinity as k tends to b. Hence the integral is equal to

$$\frac{b\sqrt{\theta(b^2+\theta)}}{b(b^2\theta)-b^2y}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]^{\frac{1}{2}}$$

$$2abc\int_{0,\,\theta_0}^{\infty}\int_0 \frac{[b(b^2+\theta)-b^2y]}{b\sqrt{\theta(b^2+\theta)}\ \sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{\{b(b^2+\theta)-b^2y\}^2}{b^2\theta\,(b^2+\theta)}s^2\right]^{\frac{1}{2}} ds\, d\theta,$$

the lower limit of θ being 0 or θ_0 according as the point is inside or outside the ellipsoid while θ_0 is defined in the same way as in §4.

Now since

$$\int_0^{\frac{a}{\beta}} \sqrt{a^2-\beta^2s^2}\, ds = \frac{\pi}{4}\,\frac{a^2}{\beta},$$ the integral will ultimately reduce to

$$\tfrac{1}{2}\pi abc\int_{0,\,\theta_0}^{\infty} \frac{d\theta}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]$$

$$=\tfrac{1}{2}P_1.$$

This is as we should expect. In the case of the complete homogeneous elliptic plate under Newtonian law of force the corresponding integral is

$$a\iint \frac{[b(b^2+\theta)-b^2y]}{\theta(b^2+\theta)\sqrt{a^2+\theta}}\, ds\, d\theta$$

$$= ab \iint \frac{[b(b^2+\theta)-b^2y]}{b\sqrt{\theta(b^2+\theta)}} \cdot \frac{1}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\, ds\, d\theta$$

while the limits of integration are defined by

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{\{b(b^2+\theta)-b^2y\}^2}{b^2\theta(b^2+\theta)}>0$$

and this by the previous method reduces to

$$ab\int_{0,\,\theta}^{\infty} \frac{d\theta}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}\right]^{\frac{1}{2}}$$

which is half the potential of the complete elliptic plate.

9. We now propose to show how the present artifice of evaluating an integral by the use of discontinuous factors may be successfully employed in determining the potential of any part of an ellipsoid and of an elliptic plate. As the procedure is the same in both cases we here deal with only the elliptic plate by way of illustration as it involves simpler calculations than the three dimensional problem which however, does not present any fresh difficulty. To avoid needless complications we assume the density function to be given by

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1}$$

remembering that the case of the more general distribution is to be treated in the usual way. Take the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\eta+k\xi+c)v}{v}\, dv+\tfrac{1}{2}$$

it is known that the value of this integral is unity or zero according as the expression $\eta+k\xi+c$ is positive or negative or in other words

according as the point (ξ, η) lies on one side or the other of the straight line

$$\eta+k\xi+c=0.$$

Hence the integral

$$P = \frac{1}{m}\frac{\Gamma(\lambda)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{r^m}\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iw)}}{(q+iw)^{\lambda}}\,dw$$

$$\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\eta+k\xi+c)v}{v}\,dv+\tfrac{1}{2}\right]$$

gives the potential of that part of an elliptic plate (a, b) of density σ which is bounded by the elliptic arc and the line $\eta+k\xi+c=0$ and encloses the portion in which the expression $\eta+k\xi+c$ is positive. This may be written as

$$P=\tfrac{1}{2}P_1+V$$

where P_1 is the potential of the complete elliptic plate of the same surface density and

$$V=\frac{1}{m}\frac{\Gamma(\lambda)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{r^m}\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iw)}}{(q+iw)^{\lambda}}\,dw$$

$$\int_{-\infty}^{\infty}\frac{\sin \eta v \cos (k\xi+c)v}{v}\,dv$$

$$+\frac{1}{m}\frac{\Gamma(\lambda)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{r^m}\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iw)}}{(q+iw)^{\lambda}}\,dw$$

$$\int_{-\infty}^{\infty}\frac{\cos \eta v \sin (k\xi+c)v}{v}\,dv$$

$$= V_1 + V_2.$$

Now making use of the integrals

$$\int_{-\infty}^{\infty} e^{-a^2x^2+2bx} \begin{matrix}\sin\\ \cos\end{matrix}\, vx\, dx = \frac{\sqrt{\pi}}{a} e^{\frac{b^2}{a^2}-\frac{v^2}{4a^2}} \begin{matrix}\sin\\ \cos\end{matrix}\, \frac{bv}{a^2}$$

it may be shown that $\qquad V_1 = W_1 + W_1',$

where

$$\begin{bmatrix} W_1 \\ W_1' \end{bmatrix} = \frac{1}{2m} \frac{\Gamma(\lambda)\sqrt{\pi}\, ab}{2m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)} \iint \frac{\theta^{-\frac{m}{2}}}{\sqrt{(a^2+\theta)(b^2+\theta)}}$$

$$\frac{\dfrac{a^2x}{a^2+\theta} \pm \dfrac{b^2y}{b^2+\theta} \pm c}{\sqrt{\dfrac{k^2a^2\theta}{a^2+\theta} + \dfrac{b^2\theta}{b^2+\theta}}} \times \left[1 - \frac{x^2}{a^2+\theta} - \frac{y^2}{b^2+\theta} - \frac{z^2}{\theta}\right.$$

$$\left. - \frac{\left(\dfrac{a^2x}{a^2+\theta} \pm \dfrac{b^2y}{b^2+\theta} \pm c\right)^2}{\dfrac{k^2a^2\theta}{a^2+\theta} + \dfrac{b^2\theta}{b^2+\theta}} \zeta^2\right]^{\lambda-\frac{m}{2}-\frac{1}{2}} d\theta\, ds;$$

wherever the double sign occurs in the integrand the upper sign is to be taken with W_1 and the lower sign with W_1'.

And similarly

$$V_2 = W_1 - W_1'.$$

Hence $\qquad V = 2W_1,$

where the limits of integration are defined by the relation

$$1 - \frac{x^2}{a^2+\theta} - \frac{y^2}{b^2+\theta} - \frac{z^2}{\theta} - \frac{\left(\dfrac{a^2x}{a^2+\theta} + \dfrac{b^2y}{b^2+\theta} + c\right)^2}{\dfrac{k^2a^2\theta}{a^2+\theta} + \dfrac{b^2\theta}{b^2+\theta}} s^2 > 0. \quad \left(\begin{matrix}1 \geq s 0 \geq \\ \theta \text{ positive}\end{matrix}\right)$$

10. Before conclusion it is necessary to recall the criticism by Kronecker[1] on Dirichlet's method of integration. He observes that however elegant Dirichlet's method may appear to be it is not altogether free from serious difficulties. The change of the order of integration and other points noted by Kronecker are not easily justifiable. These objections would stand in all the examples worked out in this paper. But it may be added that though the process of integration is unsound from the point of rigour it gives quite correctly all the known results about the potentals of uniform ellipsoid and of elliptic plates (§8). It may also be shown to give many other results in connection with problems which may be solved directly. By way of illustration we take a very simple example and propose to calculate the volume of the incomplete ellipsoid of §8 by the present method.

By the ordinary method the volume

$$=\tfrac{2}{3}\pi abc+\pi ac\int_0^k\left(1-\frac{y^2}{b^2}\right)dy$$

$$=\tfrac{2}{3}\pi abc+\pi ack\left(1-\frac{k^2}{3b^2}\right)$$

Now the use of discontinuous factor gives the volume=vol. of semi-ellipsoid $+\iiint d\xi d\eta d\zeta$ between proper limits.

$$\text{The integral}=\tfrac{1}{2}\pi\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)}dw$$

$$\frac{1}{\pi}\int_0^{\infty}\frac{\sin(k-\eta)v}{v}dv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\xi d\eta d\zeta$$

$$=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{(q+iw)}}{(q+iw)}\frac{\pi\sqrt{\pi}\,abc}{(q+iw)^{\frac{3}{2}}}\frac{e^{-\frac{b^2v^2}{4(q+iw)}}}{v}\sin vk\,dwdv.$$

[1] *l.c. ante.*

But

$$\int_0^\infty \frac{e^{-\frac{b^2v^2}{4(q+iw)}}}{v}\sin vk\,dv = \sqrt{\pi}\,\frac{k}{b}\,(q+iw)^{\frac{1}{2}}\int_0^1 e^{-(q+iw)\frac{k^2}{b^2}s^2}\,ds. \quad (\S 3).$$

Hence the integral

$$=\tfrac{1}{2}ack\int_0^1\int_{-\infty}^{\infty}\frac{e^{(q+iw)^2\left(1-\frac{k^2s^2}{b^2}\right)}}{(q+iw)^2}\,ds\,dw$$

$$=\pi ack\int_0^1\left(1-\frac{k^2}{b^2}s^2\right)ds, \text{ since } 1-\frac{k^2s^2}{b^2}>0,\ k<b,\ 1\geqq s\geqq 0$$

$$=\pi ack\left(1-\frac{k^2}{3b^2}\right)$$

which verifies the result obtained by direct calculation.

On the Wave-Equation in Ellipsoidal Co-ordinates

BY

SUDHANSUKUMAR BANERJI

1. In a previous paper[1] published in the Bulletin, it was shown that if the ellipsoidal co-ordinates ρ, θ, ϕ be defined by

$$x=a\rho \sin \theta \cos \phi,\ y=b\rho \sin \theta \sin \phi,\ z=c\rho \cos \theta,$$

then

$$\rho^n\ C_n^m\ (\theta, \phi) \text{ and } \rho^n\ S_n^m\ (\theta, \phi),$$

where

$$C_n^m\ (\theta, \phi)=\frac{(n+m)\ !}{2\pi\ n\ !\ i^m}\int_{-\pi}^{\pi} (c \cos \theta+ia \sin \theta \cos \phi \cos u$$

$$+ib \sin \theta \sin \phi \sin u)^n \cos mudu$$

and

$$S_n^m\ (\theta, \phi)=\frac{(n+m)\,!}{2\pi\ n\,!\ i^m}\int_{-\pi}^{\pi} (c \cos \theta+ia \sin \theta \cos \phi \cos u$$

$$+ib \sin \theta \sin \phi \sin u)^n \sin mudu$$

constitute a class of solutions of the Laplace's equation which are extremely suitable for problems involving an ellipsoidal boundary. The familiar method of determining by means of ordinary spherical harmonics the potential of a spherical bowl or a circular disk at any arbitrary point from the known value of the potential on the axis becomes at once available for solving similar problems for an ellipsoidal bowl or an elliptic plate. The potential of an ellipsoidal bowl or an elliptic plate can be easily obtained at any point on the axis and the potential at any other point can be at once expressed in terms of $\rho^n C_n^m\ (\theta, \phi)$, $\rho^n\ S_n^m\ (\theta, \phi)$, $\rho^{-n-1}\ \mathfrak{C}_n^m\ (\theta, \phi)$ or $\rho^{-n-1}\ \mathfrak{S}_n^m\ (\theta, \phi)$. The motion of an incompressible liquid in an ellipsoidal cup and many other potential problems can be similarly investigated. The detailed discussion

[1] Vol. X, No. 2, September, 1919.

of these problems will be given in a paper shortly to be published. In particular a method was suggested for constructing a set of solutions of the wave equation,

$$(\nabla^2+k^2)V=0,$$

in the ellipsoidal co-ordinates (ρ, θ, ϕ) in terms of these harmonics. The possibility of the solution being expressed in the form of a product $\psi_n(k\rho)\, C_n^m(\theta, \phi)$ was tacitly assumed in the method. It will be noticed in the light of the further results obtained in this paper that this assumption is not rigorously justifiable, but when the assumption has been made, it is possible to obtain an expression for $\psi_n(k\rho)$ which represents the mean value of the function on the ellipsoid ρ and that with this value for $\psi_n(k\rho)$, the quantity $\psi_n(k\rho)\, C_n^m(\theta, \phi)$ very nearly approaches a solution. An approximate treatment of the motion of a gas within a rigid ellipsoidal envelope will be given in some detail towards the end of this paper in illustration of the solutions.

2. If we put $x=ax'$, $y=by'$, $z=cz'$, then the wave equation can be written in the form

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 V}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 V}{\partial z'^2}+k^2V=0.$$

If it is assumed that this equation has a solution of the form

$$R_n\, U_n,$$

where R_n is a function of ρ only and U_n is a solution of the equation

$$\frac{1}{a^2}\frac{\partial^2 U_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 U_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 U_n}{\partial z'^2}=0,$$

then we obtain

$$\frac{1}{a^2}\frac{\partial^2(R_nU_n)}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2(R_nU_n)}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2(R_nU_n)}{\partial z'^2}+k^2R_nU_n=0,$$

that is,

$$\left[\frac{1}{a^2}\frac{\partial^2 R_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 R_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 R_n}{\partial z'^2}\right]+2\left[\frac{1}{a^2}\frac{\partial R_n}{\partial x'}\frac{\partial U_n}{\partial x'}\right.$$

$$\left.\frac{1}{b^2}\frac{\partial R_n}{\partial y'}\frac{\partial U_n}{\partial y'}+\frac{1}{c^2}\frac{\partial R_n}{\partial z'}\frac{\partial U_n}{\partial z'}\right]+k^2R_nU_n=0.$$

Now since R_n is a function of ρ only, we have

$$\frac{\partial R_n}{\partial x'}=\frac{x'}{\rho}\frac{\partial R_n}{\partial \rho},\quad \frac{\partial R_n}{\partial y'}=\frac{y'}{\rho}\frac{\partial R_n}{\partial \rho},\quad \frac{\partial R_n}{\partial z'}=\frac{z'}{\rho}\frac{\partial R_n}{\partial \rho}.$$

Therefore the above equation becomes

$$\left[\frac{1}{a^2}\frac{\partial}{\partial x'}\left(\frac{x'}{\rho}\frac{\partial R_n}{\partial \rho}\right)+\frac{1}{b^2}\frac{\partial}{\partial y'}\left(\frac{y'}{\rho}\frac{\partial R_n}{\partial \rho}\right)+\frac{1}{c^2}\frac{\partial}{\partial z'}\left(\frac{z'}{\rho}\frac{\partial R_n}{\partial \rho}\right)\right]U_n$$
$$+\frac{2}{\rho}\frac{\partial R_n}{\partial \rho}\left[\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c^2}\frac{\partial U_n}{\partial z'}\right]+k^2R_nU_n=0.$$

This can also be written in the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right)\left[\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}\right]U_n$$
$$+\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\left[2\left(\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c^2}\frac{\partial U_n}{\partial z'}\right)\right.$$
$$\left.+\,U_n\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\right]+k^2R_nU_n=0.$$

It will appear from this equation that it is not possible to separate the differential equation for R_n. But it is easy to obtain the differential equation satisfied by the mean value of R_n on any ellipsoidal surface. If we put $U_n=\rho^n C_n(\theta, \phi)$, this becomes

$$\rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right)\frac{C_n(\theta,\phi)}{p_o^2}+\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\left[2n\,D_n(\theta,\phi)\right.$$
$$\left.+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)C_n(\theta,\phi)\right]+k^2R_nC_n(\theta,\phi)=0,$$

where
$$\frac{1}{p_o^2}=\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2}$$

and

$$D_n^m(\theta,\phi)=\frac{(n+m)!}{2\pi n!\,i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u$$
$$+ib\sin\theta\sin\phi\sin u)^{n-1}$$
$$\left(\frac{\cos\theta}{c}+\frac{i\sin\theta\cos\phi\cos u}{a}+\frac{i\sin\theta\sin\phi\sin u}{b}\right)du.$$

Multiplying this equation by the conjugate function $C_n(\theta, \phi)$ and integrating we obtain

$$\rho \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\frac{\sin\theta}{p_o^2}\,d\theta d\phi$$

$$+\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\left[2n\int_0^{\pi}\int_0^{2\pi}D_n(\theta,\phi)\,C_n(\theta,\phi)\sin\theta d\theta d\phi\right.$$

$$\left.+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta d\theta d\phi\right]$$

$$+k^2R_n\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta d\theta d\phi=0.$$

Now it is easy to see that

$$(2n+3)\int_0^{\pi}\int_0^{2\pi}(C_n(\theta,\phi))^2\frac{\sin\theta}{p_o^2}\,d\theta d\phi$$

$$=2n\int_0^{\pi}\int_0^{2\pi}D_n(\theta,\phi)\,C_n(\theta,\phi)\sin\theta d\theta d\phi$$

$$+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta d\theta d\phi.$$

Hence the differential equation for R_n reduces to the form

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)+(2n+3)\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right)$$

$$k^2\frac{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta d\theta d\phi}{\displaystyle\int_0^{\pi}\int_0^{2\pi}[C_n(\theta,\phi)]^2\frac{\sin\theta}{p_o^2}\,d\theta d\phi}R_n=0.$$

If now we write

$$k^2 \frac{\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi}{\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_o^2} d\theta d\phi} = k'^2,$$

the equation can be written in the form

$$\rho \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial\rho}\right) + (2n+3)\frac{1}{\rho}\frac{\partial R_n}{\partial\rho} + k'^2 R_n = 0,$$

which has the well-known solution

$$R_n = A\psi_n(k'\rho) + B\Psi_n(k'\rho),$$

where A and B are two arbitrary constants and

$$\psi_n(\rho) = \left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \frac{\sin\rho}{\rho}, \quad \Psi_n(\rho) = \left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \frac{\cos\rho}{\rho}.$$

Also

$$\rho^n \psi_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{n+\frac{1}{2}}(\rho),$$

$$\rho^n \Psi_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{-n-\frac{1}{2}}(\rho).$$

The condition, which any two distinct solutions V, V′ of the wave equation, which themselves or their differential co-efficients in the direction of the normal vanish on the surface of the ellipsoid, must satisfy, namely

$$\int_0^1\int_0^{\pi}\int_0^{2\pi} VV'\rho^2 \sin\theta d\rho d\theta d\phi = 0,$$

is also easily seen to be satisfied by these approximate solutions.

3. One of the most interesting applications of these results is to the investigation of the motion of a gas within a rigid ellipsoidal envelope. To determine the free periods we have only to suppose that $\dfrac{\partial \psi}{\partial \rho}$ vanishes when $\rho=1$. The symmetrical vibrations in which the disturbance on each similar and similarly situated ellipsoidal surface is in the same phase will be determined by $\psi_o\ (k'\rho)$ which satisfies the equation

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi_o}{\partial \rho}\right)+\frac{3}{\rho}\cdot\frac{\partial \psi_o}{\partial \rho}+\frac{3k^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\psi_o=0.$$

Therefore

$$\psi_o=\frac{\sin\left(\dfrac{\sqrt{3}k\rho}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}\right)}{\rho}.$$

The free periods are given by

$$\frac{\sqrt{3}k\rho}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\begin{array}{l}1\cdot4303\pi,\ \ 2\cdot4590\pi\ \ 3\cdot4709\pi,\\ 4\cdot4774\pi,\ \ 5\cdot4818\pi,\ \ 6\cdot4844\pi,\ \text{etc.}\end{array}$$

The first finite root corresponds to the symmetrical vibration of lowest pitch. In the case of a higher root the vibrations in question has ellipsoidal nodes defined by the values of ρ corresponding to the inferior roots. It will be noticed that the pitch would be lower for the ellipsoidal shell than for a corresponding spherical shell obtained by putting $a=b=c=1$. The amount by which the pitch is decreased for an ellipsoidal shell of given dimensions can be easily calculated from the above formula.

4. The case of $n=1$ is perhaps the most interesting. The differential equation satisfied by ψ_1 is

$$\rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi_1}{\partial \rho}\right)+\frac{5}{\rho}\frac{\partial \psi_1}{\partial \rho}+k'^2\psi_1=0,$$

where

$$k'^2=\frac{5k^2}{\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},\quad \frac{5k^2}{\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}}\quad \text{or}\quad \frac{5k^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}}.$$

Hence the vibration at any point is given by

$$-\frac{\partial}{\partial(k'\rho)}\frac{\sin k'\rho}{k'\rho}\mathrm{U}_1,$$

where $\mathrm{U}_1=a\sin\theta\cos\phi$, $b\sin\theta\sin\phi$, or $c\cos\theta$. Hence the air sways from side to side in the directions of the three principal axes. For vibrations in the direction of the a—axis. the periods are given by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.,}$$

for vibrations in the direction of the b-axis by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.,}$$

and for the direction of the c-axis by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.}$$

5. When $n=2$, the differential equation satisfied by ψ_2 is

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_2}{\partial\rho}\right)+\frac{7}{\rho}\frac{\partial\psi_2}{\partial\rho}+k'^2\psi_2=0,$$

where $k'^2=$

$$\frac{7k^2\ [12a^4+3(b^4+c^4)-4a^2(b^2+c^2)+2b^2c^2]}{8(b^2+c^2+4a^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)[12a^4+3(b^4+c^4)-4a^2(b^2+c^2)+2b^2c^2]},$$

$$\frac{7k^2\ [12b^4+3(c^4+a^4)-4b^2(c^2+a^2)+2c^2a^2]}{8(c^2+a^2+4b^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)[12b^4+3(c^4+a^4)-4b^2(c^2+a^2)+2c^2a^2]}$$

or

$$\frac{7k^2\ [12c^4+3(a^4+b^4)-4c^2(a^2+b^2)+2a^2b^2]}{8(a^2+b^2+4c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)[12c^4+3(a^4+b^4)-4c^2(a^2+b^2)+2a^2b^2]}$$

The spherical nodes are given by

$$\tan k'\rho=\frac{k'^3\rho^3-2k'\rho}{4k'^2\rho^2-9}$$

of which the first finite solution is $k'\rho = 3{\cdot}3422$, giving a tone graver than any of the symmetrical group. The following will also be seen to be nodal surfaces

$$2x^2 - y^2 - z^2 = 0,\ 2y^2 - z^2 - x^2 = 0,\ 2z^2 - x^2 - y^2 = 0.$$

It will appear from the above results that corresponding to a single mode of vibration of the gas inside a spherical shell we get three distinct modes of vibration for the ellipsoidal shell. This result is also clear from the general expression. The periods for the nth mode are determined by k which are the roots of the equation

$$\frac{\partial}{\partial\rho}\left[\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \frac{\sin k'\rho}{\rho}\right] = 0,$$

where

$$k'^2 = k^2 \frac{\int_0^\pi\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta \, d\theta \, d\phi}{\int_0^\pi\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_0^2} d\theta \, d\phi}.$$

It is clear from the above expression that by an interchange of the letters a, b, c in the expression for $C_n(\theta, \phi)$ we get three distinct types of vibrations.

6. As we are not yet in possession of any rigorous solution of the wave-equation either in the confocal system λ, μ, ν or in the system ρ, θ, ϕ, it is thought that the approximate solutions given above may be used with advantage to elucidate some of the obscure points in the ellipsoidal problem. One advantage of the solution is that the methods already in common use for the spherical problems can be easily extended to solve analogous problems for the ellipsoidal boundary.

Further results on the harmonics $C_n^m(\theta, \phi)$, $S_n^m(\theta, \phi)$ and the solutions of the wave-equation and their applications to other physical problems will be published in due course.

On the numerical calculation of the roots of the equations $P_n^m(\mu)=0$ and $\frac{d}{d\mu} P_n^m(\mu)=0$ regarded as equations in *n*.

[Part II]

BY

BHOLANATH PAL

In my first paper,[1] "On the numerical calculation of the roots of the equations $P_n^m(\mu)=0$ and $\frac{d}{d\mu} P_n^m(\mu)=0$ regarded as equations in n," I promised to give some further tables for the values of n, for, $\theta=\frac{\pi}{4}$, $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{12}$. These tables have now been completed and are given in the following pages.

The method which I followed in calculating the roots was, as I explained in my first paper, derived from an asymptotic expansion of $P_n^m(\mu)$ as a function of n recently given by Prof. Watson in the "Transactions of the Cambridge Philosophical Society" (October, 1918). I should also point out that the expressions for n for which these functions vanish and which I have obtained from the asymptotic expansion are rapidly convergent and very convenient for numerical work.

The values of n for which $P_n^m(\mu)$ vanishes are given by

$$n=\xi+\frac{1}{\theta}\left\{a_1+a_2+a_3-\frac{a_1^3}{3}-a_1^2a_2-\dots\right\}$$

$$+\frac{1}{\theta^2}\left\{a_1a'_1+a'_1a_2+a_1a'_2+\dots\right\}$$

[1] Bulletin of the Calcutta Mathematical Society, Vol. IX, No. 2, March (1919).

where

$$\xi=\frac{\pi}{2\theta}\left\{2k-m+\frac{3}{2}-\frac{\pi}{\theta}\right\},$$

$$a_1=-\frac{C'_1}{2\xi},$$

$$a_2=\frac{C'_1C_1-3C'_2}{(2\xi)^2},$$

$$a_3=\frac{3C_2C'_1+3C'_2C_1-C_1{}^2C'_1-15C'_3}{(2\xi)^3},$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

and

$$C_1=m^2-\tfrac{1}{2},$$

$$C'_1=(m^2-\tfrac{1}{4})\cot\theta,$$

$$C_2=\tfrac{1}{6}(m^2-\tfrac{1}{2})^2-\tfrac{1}{6}(m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4})\cot^2\theta,$$

$$C'_2=\tfrac{1}{3}(m^2-\tfrac{3}{2})(m^2-\tfrac{1}{4})\cot\theta,$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

and $a'_1, a'_2 \ldots$ are written for $\dfrac{da_1}{d\xi}, \dfrac{da_2}{d\xi}, \ldots$

The roots of the equation $P_n{}^m(\mu)=0$.

TABLE I

$$\theta=\frac{\pi}{4}, \; m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	6·5	·0192	−·0014	−·0000	−·0000	·0000	·0000	6·52
2	10·5	·0119	−·0005	−·0000	−·0000	·0000	·0000	10·51
3	14·5	·0086	−·0002	−·0000	−·0000	·0000	·0000	14·51
4	18·5	·0067	−·0001	−·0000	−·0000	·0000	·0000	18·50
5	22·5	·0055	−·0001	−·0000	−·0000	·0000	·0000	22·50

The roots of the equation $P_n^m(\mu)=0$

TABLE II

$$\theta=\frac{\pi}{4},\ m=1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	4·5	—·0833	·0092	·0000	—·0015	·0003	·0001	4·40
2	8·5	—·0441	·0025	·0000	—·0002	·0000	·0000	8·44
3	12·5	—·0300	·0012	·0000	—·0000	·0000	·0000	12·46
4	16·5	—·0225	·0007	·0000	—·0000	·0000	·0000	16·47
5	20·5	—·0182	·0004	·0000	—·0000	·0000	·0000	20·47

TABLE III

$$\theta=\frac{\pi}{4},\ m=2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	2·5	—·7500	·1500	—·0190	—·2250	·0910	·0450	1·52
2	6·5	—·2884	·0221	—·0011	—·0138	·0022	·0011	6·15
3	10·5	—·1785	·0085	—·0002	—·0033	·0002	·0001	10·28
4	14·5	—·1293	·0039	—·0000	—·0010	·0000	·0000	14·34
5	18·5	—·1013	·0029	—·0000	—·0005	·0000	·0000	18·37

TABLE IV

$$\theta=\frac{\pi}{6},\ m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	10	·0216	—·0010	—·0000	—·0000	·0000	·0000	10·03
2	16	·0135	—·0004	—·0000	—·0000	·0000	·0000	16·02
3	22	·0098	—·0002	—·0000	—·0000	·0000	·0000	22·01
4	28	·0077	—·0001	—·0000	—·0000	·0000	·0000	28·01
5	34	·0063	—·0000	—·0000	—·0000	·0000	·0000	34·01

The roots of the equation $P_n^{-m}(\mu)=0$.

TABLE V

$\theta=\frac{\pi}{6},\ m=1$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	7	−·0927	·0066	·0000	−·0013	·0001	·0000	6·83
2	13	−·0499	·0019	·0000	−·0001	·0000	·0000	12·91
3	19	−·0341	·0009	·0000	−·0000	·0000	·0000	18·93
4	25	−·0259	·0005	·0000	−·0000	·0000	·0000	24·95
5	31	−·0209	·0003	·0000	−·0000	·0000	·0000	30·96

TABLE VI

$\theta=\frac{\pi}{6},\ m=2$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	4	−·8118	·1014	−·1231	−·1652	·0413	·0206	2·26
2	10	−·3247	·0162	−·0082	−·0105	·0011	·0005	9·39
3	16	−·2027	·0063	−·0031	−·0025	·0001	·0000	15·62
4	22	−·1476	·0033	−·0012	−·0009	·0000	·0000	21·72
5	28	−·1159	·0020	−·0007	−·0004	·0000	·0000	27·78

TABLE VII

$\theta=\frac{\pi}{12},\ m=0$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	20·5	·0227	−·0005	−·0000	−·0000	·0000	·0000	20·60
2	32·5	·0143	−·0002	−·0000	−·0000	·0000	·0000	32·55
3	44·5	·0104	−·0001	−·0000	−·0000	·0000	·0000	44·53
4	56·5	·0082	−·0000	−·0000	−·0000	·0000	·0000	56·53
5	68·5	·0068	−·0000	−·0000	−·0000	·0000	·0000	68·52

The roots of the equation $P_n^m(\mu)=0$

TABLE VIII

$$\theta=\frac{\pi}{12},\ m=1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	14·5	—·0965	·0033	·0003	—·0006	·0000	·0000	14·14
2	26·5	—·0528	·0009	·0000	—·0000	·0000	·0000	26·30
3	38·5	—·0363	·0004	·0000	—·0000	·0000	·0000	38·36
4	50·5	—·0277	·0002	·0000	—·0000	·0000	·0000	50·39
5	62·5	—·0223	·0001	·0000	—·0000	·0000	·0000	62·41

... ...

TABLE IX

$$\theta=\frac{\pi}{12},\ m=2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	8·5	—·8221	·0484	—·0723	—·0877	·0103	·0051	4·77
2	20·5	—·3414	·0083	—·0061	—·0057	·0003	·0001	19·21
3	32·5	—·2153	·0033	—·0023	—·0015	·0000	·0000	31·67
4	44·5	—·1572	·0017	—·0006	—·0005	·0000	·0000	43·90
5	56·5	—·1238	·0010	—·0000	—·0000	·0000	·0000	56·03

The values of n which make $\frac{d}{d\mu}P_n^m(\mu)$ vanish are given by

$$n=\xi+\frac{1}{\theta}\left\{a_1+a_2+a_3-\frac{a_1^3}{3}-a_1^2a_2-\dots\right\}+\frac{1}{\theta^2}\{a_1a'_1+a'_1a_2$$

$$+a_1a'_2+\dots\}$$

where

$$\xi=\frac{\pi}{2\theta}\left(2k-m+\frac{1}{2} - \frac{\theta}{\pi} \right),$$

$$a_1= \frac{a_o}{a'_o}$$

$$a_2=\left(\frac{a_1}{a'_o} - \frac{a_o a'_1}{a'^2_o} \right)\frac{1}{2\xi},$$

$$a_3=\left(\frac{3a_2}{a'_o} - \frac{a_1 a'_1}{a'^2_o} - \frac{3a'_2 a_o}{a'^2_o} + \frac{a'^2_1 a_o}{a'^3_o} \right) \frac{1}{(2\xi)^2},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and

$$a_o=-\tfrac{1}{2} \cot \theta,$$

$$a'_o=(\xi+\tfrac{1}{2}),$$

$$a_1=-(\xi+\tfrac{1}{2}) (m^2-\tfrac{1}{4}) \cot \theta-\tfrac{1}{2} (m^2-\tfrac{1}{2}) \cot \theta,$$

$$a'_1=(\xi+\tfrac{1}{2}) (m^2-\tfrac{1}{2})-(m^2-\tfrac{1}{4})-\tfrac{3}{2} (m^2-\tfrac{1}{4}) \text{cat}^2\theta,$$

$$a_2=-\tfrac{1}{3} (\xi+\tfrac{1}{2}) (m^2-\tfrac{3}{2}) (m^2-\tfrac{1}{4}) \cot \theta+\tfrac{1}{3} (m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4}) \cot \theta$$

$$-\tfrac{1}{12} (m^2-\tfrac{1}{2})^2 \cot \theta+\tfrac{5}{12} (m^2-\tfrac{9}{4}) (m^2-\tfrac{1}{4}) \cot^3\theta,$$

$$a'_2=\tfrac{1}{6} (\xi+\tfrac{1}{2}) (m^2-\tfrac{1}{2})^2-\tfrac{1}{3} (m^2-\tfrac{3}{2}) (m^2-\tfrac{1}{4})$$

$$-\tfrac{1}{6} (\xi+\tfrac{1}{2}) (m^2-\tfrac{1}{4}) (m^2-\tfrac{9}{4}) \cot^2\theta-\tfrac{1}{12}(m^2-\tfrac{3}{2}) (m^2-\tfrac{1}{4}) \cot^2\theta,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and $a'_1, a'_2,\ldots$ are written for $\frac{da_1}{d\xi}, \frac{da_2}{d\xi} \ldots \ldots$

The roots of the equation $\frac{d}{d\mu} P_n^m(\mu)=0$

TABLE X

$\theta=\frac{\pi}{6}, m=0$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	7	—·1154	·0309	—·0003	—·0018	·0006	·0005	6·83
2	13	—·0641	·0166	—·0002	—·0002	·0001	·0001	12·89
3	19	—·0444	·0113	—·0000	—·0001	·0000	0000	18·93
4	25	—·0339	·0086	—·0000	—·0000	·0000	·0000	24·95
5	31	—·0274	·0069	—·0000	—·0000	·0000	·0000	30·96

TABLE XI

$\theta=\frac{\pi}{6}, m=-1$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	10	—·0824	—·0665	·0014	—·0006	—·0005	—·0004	9·71
2	16	—·0524	—·0410	·0005	—·0002	—·0001	—·0001	15·82
3	22	—·0385	—·0296	·0003	—·0000	—·0000	—·0000	21·87
4	28	—·0303	—·0232	·0001	—·0000	—·0000	—·0000	27·89
5	34	—·0251	—·0191	·0000	—·0000	—·0000	—·0000	33·91

TABLE XII

$\theta=\frac{\pi}{6}, m=-2$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	13	—·0641	—·2535	—·0000	—·0002	—·0015	—·0013	12·38
2	19	—·0444	—·1733	·0015	—·0001	—·0005	—·0004	18·58
3	25	—·0339	—·1303	·0012	—·0000	—·0003	—·0002	24·68
4	31	—·0274	—·1047	·0008	—·0000	—·0001	—·0001	30·74
5	37	—·0251	—·0877	·0000	—·0000	—·0000	—·0000	36·78

The roots of the equation $\frac{d}{d\mu} P_n^m(\mu)=0$

TABLE XIII

$\theta=\frac{\pi}{12}, m=0$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	14·5	—·1244	·0336	·0008	—·0009	·0003	·0002	14·12
2	26·5	—·0691	·0178	·0000	—·0001	·0000	·0000	26·29
3	38·5	—·0504	·0121	·0000	—·0000	·0000	·0000	38·35
4	50·5	—·0365	·0092	·0000	—·0000	·0000	·0000	50·39
5	62·5	—·0296	·0074	·0000	—·0000	·0000	·0000	62·41

TABLE XIV

$\theta=\frac{\pi}{12}, m=-1$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	20·5	—·0888	—·0698	—·0005	—·0003	—·0003	—·0002	19·88
2	32·5	—·0565	—·0434	—·0000	—·0000	—·0001	—·0000	32·11
3	44·5	—·0414	—·0315	—·0000	—·0000	—·0000	—·0000	44·22
4	56·5	—·0327	—·0271	—·0000	—·0000	—·0000	—·0000	56·28
5	68·5	—·0270	—·0204	—·0000	—·0000	—·0000	—·0000	68·32

TABLE XV

$\theta=\frac{\pi}{12}, m=-2$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'^1a_2	n
1	26·5	—·0691	—·2676	—·0051	—·0001	—·0007	—·0006	25·06
2	38·5	—·0504	—·1829	—·0009	—·0000	—·0002	—·0002	37·58
3	50·5	—·0365	—·1388	—·0000	—·0000	—·0000	—·0000	49·83
4	62·6	—·0296	—·1119	—·0000	—·0000	—·0000	—·0000	61·96
5	74·5	—·0248	—·0939	—·0000	—·0000	—·0000	—·0000	74·03

Notes

We have received for publication from Professor Frechét of the University of Strassbourg the following courses of instruction to be delivered in the University of Strassbourg in the next academic year. The University has been informally reopened and reorganised this year and Professor Frechét desires the attendance of some students or scholars from India, graduates being preferred.

Programme des cours et des conférences de Mathématiques l'annee scolaire 1919-20.

IER SEMESTRE.

Mathématiques générales et mathématiques préparatoires :

M. N., professeur	...	3 cours par semaine
M. DARMOIS, professeur	...	2 conférences par semaine

Calcul différentiel et intégral :

M. VALIRON, professuer	...	3 cours par semaine
M. ANTOINE, maître de conférence.		2 conférences par semaine

Mécanique rationnelle :

M. VILLAT, professeur	...	3 cours par semaine
M. VERONNET, chargé de conférences		2 conférences mar semaine

Astronomie :

M. ESCLANGON, professeur	...	2 cours par semaine

*Analyse supérieure :**

M. FRECHET, professeur (Calcul fonctionnel).		2 cours par semaine
(Fonctions d' aproximation)	...	1 cours par semaine

* Les cours dont les titres sont suivis d'une astérisque portent sur des sujets variables chaque année et s'adressent aux étudiands avancés.

2IÈME SEMESTRE.

Mathématiques générales :

M. N., Professeur ...	1 cours par semaine
M. DARMOIS, maître de conférences.	2 conférences par semaine

Calcul différentiel et intégral :

M. VALIRON, professeur ...	1 cours par semaine
M. ANTOINE, maître de conférence.	2 conférences par semaine

Mécanique rationnelle :

M. VILLAT, professeur ...	1 cours par semaine
M. VERONNET, chargé de conférence.	2 conférences par semaine

Astronomie :

M. ESCLANGON, professeur ...	2 cours par semaine
M. DANJON, astronome-adjoint	Travaux pratiques à L'observatoire

*Analyse supérieure :**

M. FRECHET, professeur (calcul Fonctionnel).	3 cours par semaine

*Géométrie supérieure :**

M. N., professeur (Déformation des Surfaces).	2 cours par semaine

*Theorie des fonctions :**

M. VALIRON, professeur ; Fonctions entiéres.	2 cours par semaine
M. VILLAT, professeur : Fonctions elliptiques (avec application a la Physique Mathématiques).	2 cours par semaine

* Les cours dont les titres sont suivis d'une astérisque portent sur des sujets variables chaque annéę et s'adressent aux étudiands avancés.

IER ET 2IÉME SEMESTRE.

Préparation, à l'enseignement.

(sous la direction de M. VILLAT, professeur)

Mathématiques spéciales :

M. VILLAT, professeur ... 1 conférence par semaine

Mathématiques élémentaires :

M. N. .. 1 conférence par semaine

Calcul différentiel et intégral :

M .ANTOINE, maître de conférence. 1 conférence par semaine

Mécanique rationnelle :

M. DARMOIS, maître de conférence. 1 conférence par semaine

Travaux pratiques de Mathématiques :

M. N., Directeur du laboratoire de Mathématiques.
M. N., Préparateur de Mathématiques.
L'horaire sera rétabli suivant le nombre des étudiants inscrits.

Institut de Mathèmatiques :

M. FRECHET, Directeur.

L'horaire des colloques mathématiques (destinés à encourager les recherches originales) sera établi ultérieurement suivant le nombres des chercheurs inscrits.

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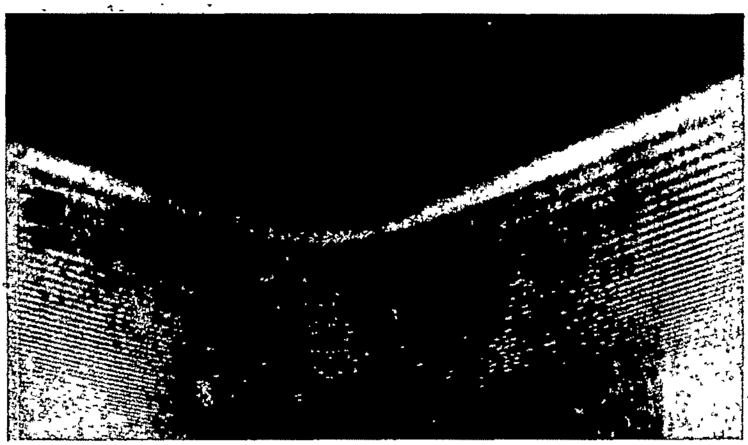
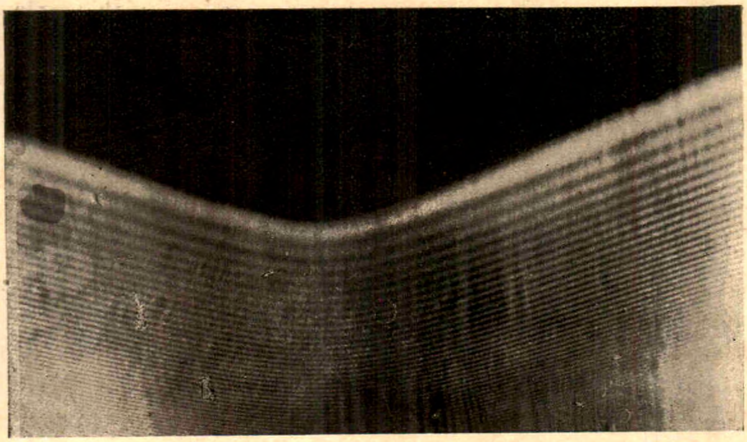
BULLETIN OF THE CALCUTTA
MATHEMATICAL SOCIETY, VOL. X, No 4.



Illustrating the scattering of light by a perfectly reflecting cone.

To Face Page 207

BULLETIN OF THE CALCUTTA
MATHEMATICAL SOCIETY, VOL. X, No. 4.



Illustrating the scattering of light by a perfectly reflecting cone.

On the Diffration of Light by a Transparent Wedge

BY

SUDHANSUKUMAR BANERJI

[*Read April 6th, 1919*]

1. Although the problem of the diffraction of light by a perfectly reflecting and perfectly absorbing wedge has been very successfully treated by Sommerfeld and other writers including Jackson[1] Macdonald,[2] Reiche,[3] Wiegrefe,[4] Bromwich[5] and Whipple[6] by the use of a certain type of contour integral, the problem of the diffraction of light by a transparent wedge, when the velocity of radiation in the body differs by a finite amount from that in the surrounding medium does not appear to have received so much attention. There are in fact not many cases in which the influence of the material properties of the obstacle has been taken into account in the mathematical treatment of a diffraction problem although the necessity of doing this has been very clearly indicated by some of the experimental results. It was therefore considered that the treatment of the problem of diffraction of light by a dielectric wedge given in this paper may not probably be altogether devoid of interest. After this paper was read, Prof. Carslaw has published an important paper on the diffraction of light by a perfectly conducting wedge of any angle in the *Proceedings of the London Mathematical Society*, December, 1919. The present paper had to be re-written in view of the results obtained by Prof. Carslaw in this paper.

2. Let the edge of the wedge be chosen as the axis of z and let r, θ, z be the cylindrical co-ordinates of a point so that the faces of the wedge are given by $\theta=0$ and $\theta=\beta$ and that the space occupied by it is that between $\theta=\beta$ and $\theta=2\pi$. Taking the case in which the electric force

[1] Jackson, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1 (1904), p. 393.

[2] Macdonald, *Electric Waves*, Appendix D (1902); *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 14 (1915), p. 410.

[3] Reiche, *Ann. d. Phys.*, Bd. 37 (1912), p. 131.

[4] Wiegrefe, *Ann. d. Phys.*, (4), Bd. 39 (1912), p. 449.

[5] Bromwich, *Proc. Lond. Math. Soc.*, Vol. 14 (1915), p. 450.

[6] Whipple, *Proc. Lond. Math. Soc.*, Vol. 16 (1917), p. 94.

in the incident wave is parallel to the edge of the wedge, it is easy to see that the electric force in the total disturbance outside the wedge and that in the disturbance which enters into the body of the wedge must both be parallel to the edge of the wedge. Let Z_0 denote the electric force in the incident wave, then

$$Z_0 = e^{ik_1[r\cos(\theta-\theta')+V_1 t]},$$

which represents a set of plane waves coming from the direction θ'. Prof. Carslaw[1] has transformed this expression into the integral

$$\frac{1}{2\pi}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]}\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha,$$

over the path (A) in the α—plane (see the figure in his paper). He has also obtained the following expression which is a finite and continuous solution of the wave equation and of period $2n\pi$, n being any positive integer,

$$Z = \frac{1}{2n\pi}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]}\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha,$$

the integral being taken over the same path (A) in the α—plane. This integral has the property that as $r \longrightarrow \infty$, $Z \longrightarrow Z_0$, when $|\theta-\theta'| < \pi$, and $Z \longrightarrow 0$ when $\pi < |\theta-\theta'| < 2n\pi$. A solution of period 2β is similarly obtained in the form

$$\frac{1}{2\beta}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]}\frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta}-e^{i\pi\theta'/\beta}}\,d\alpha,$$

over the same path (A) in the α—plane. These periodic solutions of the wave equation have been used by Prof. Carslaw in the discussion he has recently given of the problem of the diffraction of waves by a perfectly conducting wedge without making any use of Riemann's surfaces.

A more general solution of the wave equation of period 2β can be obtained by starting from the integral

$$\frac{1}{2n\pi}\int f[r\cos(\alpha-\theta)+V_1 t]\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha,$$

[1] See his paper on "Multiform Solutions" in the *Proc. Lond. Math. Soc.*, Ser. 1, Vol. 30 (1898), p. 121.

over the path (A) in the a—plane, where $f\,[r\cos(\theta-\theta')+V_1t]$ is an arbitrary continuous solution of the wave equation.

$$\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}-\frac{1}{V_1^2}\frac{\partial^2 Z}{\partial t^2}=0.$$

By making n tend to infinity and by proceeding as in §3 of Prof. Carslaw's paper already quoted, we can easily reduce the integral to the form

$$\frac{1}{2i\pi}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1t)\,\frac{2\zeta}{\zeta^2-(\theta-\theta')^2}\,d\zeta,$$

where $2\pi>\gamma>\pi$, $0>\gamma'>-\pi$ and by adding to this the solutions of the same type which correspond to the directions

$$\theta'\pm2\beta,\ \theta'\pm4\beta,\ldots,$$

we obtain the following solution of period 2β,

$$\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1t)\,\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')}\,d\zeta.$$

Similarly by adding the solutions which correspond to waves in the directions

$$-\theta'\pm2\beta,\ -\theta'\pm4\beta,\ldots,$$

we obtain the solution

$$\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1t)\,\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\,d\zeta,$$

where $$2\pi>\gamma>\pi,\ 0>\gamma'>-\pi.$$

3. If now Z_1 denotes the electric force in the disturbance outside the wedge and Z_2 that in the disturbance inside the wedge, then since at the boundary between two media, the tangential electric force must be continuous, the following condition must be satisfied on the two faces of the wedge :

$$Z_1 = Z_2, \text{ when } \theta = 0 \text{ and } \theta = \beta,$$

that is,

$$Z_1 - Z_2 = 0, \text{ when } \theta = 0 \text{ and } \theta = \beta.$$

We can therefore assume the following expressions for Z_1 and Z_2 :

$$Z_1 = \frac{Ai}{2\beta} \int_{\infty i + \gamma}^{\infty i + \gamma'} e^{ik_1 (r \cos \zeta + V_1 t)} \left[\frac{\sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta - \theta')} - \frac{\sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta + \theta')} \right] d\zeta,$$

$$+ \frac{i}{2\beta} \int_{\infty i + \gamma}^{\infty i + \gamma'} f(r \cos \zeta + V_1 t) \left[\frac{C \sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta - \theta')} + \frac{D \sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta + \theta')} \right] d\zeta,$$

$$Z_2 = \frac{i}{2\beta} \int_{\infty i + \gamma}^{\infty i + \gamma'} F(r \cos \zeta + V_2 t) \left[\frac{E \sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta - \theta')} + \frac{F \sin \frac{\pi \zeta}{\beta}}{\cos \frac{\pi \zeta}{\beta} - \cos \frac{\pi}{\beta} (\theta + \theta')} \right] d\zeta,$$

V_1, V_2 denoting the velocities of propagation in the two media.

If now we suppose that the two arbitrary functions F and f satisfy the equation

$$F(r\cos\zeta+V_2t)=f(r\cos\zeta+V_1t),$$

for all values of $r\cos\zeta$ and t, then the condition

$$Z_1-Z_2=0, \text{ when } \theta=0 \text{ and } \theta=\beta,$$

is satisfied if $C=F$ and $D=E$. The alternative solution $C=-D$ and $E=-F$ is obviously inadmissible as we are then led to the condition that Z_1 and Z_2 separately vanish on the two faces of the wedge $\theta=0$ and $\theta=\beta$. It is also easy to see that if we choose the two arbitrary functions to have the forms

$$f(r\cos\zeta+V_1t)=e^{ik_1(r\cos\zeta+V_1t)},$$

and $$F(r\cos\zeta+V_2t)=e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\frac{k_1-k_2}{k_2}\left[ik_2(r\cos\zeta+V_2t)\right]e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\frac{(k_1-k_2)^2}{2!\,k_2^2}\left[ik_2(r\cos\zeta+V_2t)\right]^2e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\ldots,$$

then the two functions are very nearly identical for all values of $r\cos\zeta$ and t, provided that

$$k_1V_1=k_2V_2$$

and k_1, k_2 differ only by a small quantity. When k_1 and k_2 differ by a finite quantity, the form of the function F can be obtained by the method of successive approximations.

We thus obtain the following expressions for Z_1 and Z_2:

$$Z_1 = \frac{Ai}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)} \left[\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} - \frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta,$$

$$+ \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)} \left[\frac{C\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{D\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta,$$

$$Z_2 = \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} F(r\cos\zeta+V_2 t) \left[\frac{D\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{C\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta.$$

4. If (R_1, Θ_1) denote the components of magnetic force in the directions r and θ in the disturbance outside the wedge and (R_2, Θ_2) those in the directions r, θ in the disturbance inside the wedge, we have

$$-i\mu_1 R_1 = \frac{1}{r}\frac{\partial Z_1}{\partial\theta}, \quad i\mu_1\Theta_1 = \frac{\partial Z_1}{\partial r},$$

$$-i\mu_2 R_2 = \frac{1}{r}\frac{\partial Z_2}{\partial\theta}, \quad i\mu_2\Theta_2 = \frac{\partial Z_2}{\partial r},$$

μ_1, μ_2 being the magnetic permeabilities of the two media.

Since at the boundary of two media, the tangential components of the magnetic force must be continuous, we have

$$R_1-R_2=0, \text{ when } \theta=0 \text{ and } \theta=\beta.$$

This gives

$$\frac{2A}{\mu_1}+\frac{C-D}{\mu_1}+\frac{C-D}{\mu_2}=0,$$

that is, $$C-D=-2A\mu_2/(\mu_1+\mu_2).$$

The condition that at the boundary between two media the normal magnetic induction must be continuous, that is to say,

$$\mu_1\Theta_1-\mu_2\Theta_2=0, \text{ when } \theta=0 \text{ and } \theta=\beta,$$

is easily seen to be automatically satisfied.

Also as r tends to infinity

$$Z_1 \longrightarrow e^{ik_1[r\cos(\theta-\theta')+V_1t]}.$$

It will thus appear that the above expressions for Z_1 and Z_2 satisfy all the conditions of the problem and it will also be seen that we have used one more constant than are actully required for the solution of the problem. We can therefore impose one relation between C and D consistent with the conditions of the problem. We can therefore assume

$$C+D=A-1.$$

Thus we get

$$C=\tfrac{1}{2}(A-1)-\frac{A\mu_2}{\mu_1+\mu_2},$$

$$D=\tfrac{1}{2}(A-1)+\frac{A\mu_2}{\mu_1+\mu_2}.$$

We thus finally obtain the following expressions for Z_1 and Z_2

$$Z_1 = \frac{Ai}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1t)} \left[\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} - \frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta,$$

$$+ \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1t)} \left[\frac{\left\{\frac{1}{2}(A-1) - \frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{\left\{\frac{1}{2}(A-1) + \frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta,$$

$$Z_2 = \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} F(r\cos\zeta+V_2t) \left[\frac{\left\{\frac{1}{2}(A-1) + \frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{\left\{\frac{1}{2}(A-1) - \frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta.$$

For the absolute determination of A we have to equate the energy contained in the primary wave with the energy contained in the disturbance functions Z_1 and Z_2. This will give a linear equation for the determination of A.

When the wedge is perfectly conducting, we have $Z_2=0$ at all points inside the wedge This gives $C=0$, $D=0$ and $A=1$. The expression for Z_1 then agrees with the well-krown solution for a perfectly conducting wedge.

In the particular case when the dielectric properties of the two media differ very little from each other, the expression for Z_2 becomes considerably simplified, as we have already seen, for we can then neglect all powers of k_1-k_2 beyond the first.

The modification to be made in the treatment given above for the associated problem when the magnetic force in the incident wave is parallel to the edge of the wedge can be easily obtained.

On a geometrical treatment of the scattering of light by a perfectly reflecting cone

[With a Plate]

BY

ABANIBHUSAN DATTA

[*Read August 31st, 1919*]

§ 1. The present paper is an attempt to explain the form of the diffraction fringes observed when a perfectly reflecting cone is exposed to a point source of light by a geometrical method which though not very accurate from the point of view of the wave theory is still found to be capable of explaining the phenomena. If the diffraction fringes are photographed on a plate placed behind the cone, it is found that the figures are approximately a series of hyperbolas having their asymptotes parallel to the two lines which are the projection of the cone on the plate. This will be evident from the plate annexed to this paper which has been kindly lent by Prof. C. V. Raman to the present writer.

A rigorous mathematical theory of the problem of the scattering of sound waves by a cone has been given by Prof. Carslaw.[1] But the solution obtained by him does not appear to be easily amenable to numerical calculation.

I wish to express my obligation to Prof. S. K. Banerji for the many valuable suggestions, I have obtained from him regarding this problem.

§ 2. If A be the source of light, a ray falling on the surface of the cone at the point B, will be reflected in the plane containing the incident ray and the normal and making an equal angle with this

[1] *Math. Annalen*, Vol. 75 (1914), p. 143.

normal as the incident ray according to the law of geometrical optics. Whether a point on the reflected ray coincides with a maximum or minimum intensity will be determined by the path difference (AB + BP − AP). If we put this quantity $=n\lambda$, where λ is the wave length and n is any integer and if we determine the locus of P, the locus will correspond to a curve of maximum or minimum intensity. This locus can be obtained by a simple geometrical construction. If we conceive a set of confocal ellipsoids of revolution constructed having the points A and P as foci and AP as the axis of revolution, one and only one of this set will touch the cone at the point B. Conversely, it follows that if we construct an ellipsoid of revolution about an axis AP having A as one of its foci and touching the cone at the point B, then P will be its other focus. Now if k be the major axis of such an ellipsoid of revolution, and e the eccentricity, the expression for the path difference can be written in the form $k-ke=n\lambda$. It is therefore evident that if we construct all the possible ellipsoids of revolution which can touch the cone, having one point A which is the source of light as a fixed focus and a variable second focus P, the locus of the variable focus P will be a curve either of maximum or minimum intensity.

§ 3. Let the vertex of the cone be taken as the origin and the plane determined by the given focus and the axis of the cone as the xy-plane. Then the equation of the cone can be written in the form

$$fx^2+y^2+z^2=0 \qquad \dots \quad (1)$$

Let the co-ordinates of the given focus be $(\alpha, \beta, 0)$ and those of the variable focus be (a, b, c). Hence the equation of the ellipsoid is

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}+\sqrt{(x-\alpha)^2+(y-\beta)^2+z^2}=k.$$

On rationalization, this is equivalent to

$$4x^2\{(a-\alpha)^2-k^2\}+4y^2\{(b-\beta)^2-k^2\}+4z^2(c^2-k^2)+8yz(b-\beta)c$$

$$+8zx(a-\alpha)c+8xy(a-\alpha)(b-\beta)+4\{x(a-\alpha)+y(b-\beta)+zc\}$$

$$\{k^2+(\alpha^2+\beta^2)-(a^2+b^2+c^2)\}+8k^2(\alpha x+\beta y)+[\{k^2-(\alpha^2+\beta^2)$$

$$-(a^2+b^2+c^2)\}^2-4k^2(\alpha^2+\beta^2)]=0. \qquad \dots \quad (2)$$

Let the discriminant of (2) be Δ, and A, B, C its leading first minors. Then

$$\Delta = -64k^6\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}^2 \quad \ldots \quad (3)$$

$$A = 16k^4\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2-4a\alpha\}.$$

For brevity let us put $N = k^2-(a-\alpha)^2-(b-\beta)^2-c^2$

Hence

$$\left.\begin{aligned} A &= 16k^4N(N-4a\alpha) \\ B &= 16k^4N(N-4b\beta) \\ C &= 16k^4N^2. \end{aligned}\right\} \quad \ldots \quad (4)$$

The second minors (a, b, c, d, l, m, n on the left-hand-side having their usual meanings as constituents of Δ) are given by

$$\left.\begin{aligned} ad-l^2 &= -4k^2[N(N-4b\beta)-4\beta^2c^2] \\ bd-m^2 &= -4k^2[N(N-4a\alpha)-4\alpha^2c^2] \\ cd-n^2 &= -4k^2[N(N-4a\alpha-4b\beta)-4(a\beta-b\alpha)^2] \end{aligned}\right\} \quad \ldots \quad (5)$$

§. 4 The condition that (1) should touch (2) is that the discriminant of the biquadratic

$$\lambda^4\Delta+\lambda^3\Theta+\lambda^2\Phi+\lambda\Theta'+\Delta'=0 \quad \ldots \quad (6)$$

should vanish, where, on calculation,

$$\left.\begin{aligned} &\Delta=\Delta, \\ &\Theta = 16k^4N[(f+2)N-4fa\alpha-4b\beta], \\ &\Phi = -4k^2[2(f+1)N^2-4(2fa\alpha+fb\beta+b\beta)N-4c^2(f\alpha^2+\beta^2)-4f(a\beta-b\alpha)^2] \\ &\Theta' = f[\{k^2+(\alpha^2+\beta^2)-(a^2+b^2+c^2)\}^2-4k^2(\alpha^2+\beta^2)], \\ &\Delta'=0. \end{aligned}\right\} \quad (7)$$

The discriminant of (6) on simplification reduces to

$$\Theta'^2\{\Theta^2\Phi^2-4\Theta^3\Theta'+18\Delta\Theta\Phi\Theta'-27\Delta^2\Theta'^2-4\Delta\Phi^3\} \quad \ldots \quad (8)$$

But since $\Theta' \neq 0$ (that being the condition that the ellipsoid passes through the vertex of the cone which is a particular case), the condition reduces to vanishing of D, the expression within the bracket in (8).

§ 5. Now $$k-ke=n\lambda$$

or $$k=n\lambda+ke=n\lambda+\sqrt{(a-\alpha)^2+(b-\beta)^2+c^2} \quad \ldots \quad (9)$$

Hence the locus of (a, b, c), the variable focus, is obtained by eliminating k between (9) and D. Neglecting the common factor $4^6k^{12}\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}^2$ in D, we get the locus to be a surface of the sixth degree. On such a surface therefore, the light is either minimum or maximum.

§ 6. Section of the locus by the plane y=constant is obtained by putting $b=d$ (say) in D. From (9), putting $b=d$,

$$k^2-(a-\alpha)^2-(d-\beta)^2-c^2=n^2\lambda^2+2n\lambda\sqrt{(a-\alpha)^2+(d-\beta)^2+c^2}$$

$$=\text{M (say)}. \quad \ldots \quad (10)$$

Eliminating k by means of (10), leaving out the common factor we get

$$\Delta=-64\text{M}^3,\ \Theta=16\text{M}\{\text{M}(f+2)-4fa\alpha-4d\beta\},$$

$$\Phi=-4\{\text{M}^2(2f+1)-4\text{M}(2fa\alpha+fd\beta+d\beta)-4c^2(f\alpha^2+\beta^2)-4f(a\beta-d\alpha)^2\},$$

$$\Theta'=f[\text{M}^2-(4d\beta+4a\alpha)\text{M}-4(a\beta-d\alpha)^2-4c^2(\alpha^2+\beta^2)].$$

Now, when the focus A is at infinity (*i.e.*, when the rays of light are parallel), α, β, and N are all infinite and we can put for them $l\infty$, $m\infty$ and $2n\lambda.\infty$ respectively. Then leaving out the common factor in D and dividing by $(\infty)^6$, we get the locus in the form

$$\text{D}_1=\Theta_1{}^2\Phi_1{}^2-4\Phi_1{}^3-8n\lambda\Theta_1{}^3\Theta'_1+36n\lambda\Theta_1\Phi_1\Theta'_1-108n^2\lambda^2\Theta_1{}^2=0, \quad (11)$$

where

$$\Theta_1 = 2n\lambda(f+2) - 4flx - 4dm,$$

$$\Phi_1 = 4n^2\lambda^2(2f+1) - 8n\lambda(2flx + fdm + dm) - 4z^2(fl^2+m^2) - 4f(xm-dl)^2,$$

$$\Theta'_1 = f[4n^2\lambda^2 - 8n\lambda(dm+lx) - 4(xm-dl)^2 - 4z^2].$$

By giving different integral values to n, we get the different curves corresponding to maximum or minimum intensity.

§ 7. Since $n\lambda$ is in general a very small quantity, the last three terms in (11) can be neglected in comparison with the first two and the section of the locus by the plane y=constant is therefore approximately given by

$$\Theta_1^2\Phi_1^2 - 4\Phi_1^3 = 0,$$

or

$$\Phi_1^2(\Theta_1^2 - 4\Phi_1) = 0$$

The second degree terms in $(\Theta_1^2 - 4\Phi_1)$ are

$$16(fl^2+m^2)(fx^2+z^2).$$

Hence $\Theta_1^2 - 4\Phi_1 = 0$ represents a set of hyperbolas, the directions of the asymptotes of which are $fx^2+z^2=0$ which are the same as those given by the section of the cone by the plane.

It appears that the curves represented by $\Phi_1^2 = 0$ do not belong to the particular phenomena discussed in this paper.

§ 8. Now if the light is supposed to come from a direction making an angle of $60°$ with the axis and if the cone is supposed to have a semivertical angle of $45°$, then $f=-1$, $l=\frac{1}{2}$, $m=\sqrt{\frac{3}{2}}$. Let $\lambda = \cdot 0006$mm, $n=1$, $d=10{,}000\ kn\lambda$ where $k=\sqrt{3}$. For these substitutions,

$$\Theta_1 = 2x - 35\cdot 9988$$

$$\Phi_1 = 3x^2 - 2z^2 - 35\cdot 9952x + 107\cdot 9999.$$

Hence $\Theta_1^2 - 4\Phi_1 \equiv x^2 - z^2 - 108 = 0$ ... (12)

and the direction of the asymptotes of this are given by

$$x^2 - z^2 = 0.$$

It is to be noted that by giving to n, the values 2, 3, 4 etc. in succession, we shall get a series of hyperbolas which correspond to the curves of maximum or minimum intensity.

A Note on the Deformation of Surfaces

BY

BHUPATIMOHAN SEN

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It is usually stated in the standard works on Differential Geometry that the problem of deformation depends on a partial differential equation of the second order of the Monge—Ampere type. Given the Cartesian co-ordinates of a surface in terms of two parameters (u, v) it has been proved that each of the co-ordinates satisfies the differential equation

$$(x_{11}-x_1\Gamma-x_2\Delta)(x_{22}-x_1\Gamma''-x_2\Delta'')-(x_{12}-x_1\Gamma'-x_2\Delta')$$
$$=k\{V^2-(Gx_1{}^2-2Fx_1x_2+Ex_2{}^2)\}. \quad \ldots \quad (1)$$

[Forsyth, Diff. Geom., p. 363]

When one of the co-ordinates, say x, has been determined by this equation, the others can be determined by quadrature from the equations

$$\left.\begin{aligned} x_1{}^2+y_1{}^2+z_1{}^2&=E,\\ x_2{}^2+y_2{}^2+z_2{}^2&=G,\\ x_1x_2+y_1y_2+z_1z_2&=F. \end{aligned}\right\} \quad \ldots \quad (2)$$

But there is nothing to shew that the values of y and z thus obtained after the substitution of the value of x will satisfy the Monge-Ampere equation which must be satisfied by the Cartesian co-ordinates of all surfaces having the same E, F and G.

The orthodox solution of the problem of deformation is therefore incomplete. In fact it is of doubtful value. The equation of deformation is a necessary but not a sufficient condition as will now be shown.

The real solution will be obtained by eliminating two of the three independent variables in the simultaneous equations

$$x_1^2+y_1^2+z_1^2=\mathrm{E},$$
$$x_1x_2+y_1y_2+z_1z_2=\mathrm{F},$$
$$x_2^2+y_2^2+z_2^2=\mathrm{G}.$$

But the straightforward elimination is impracticable.

Let us introduce the new independent variables D, D′, D″ given by

$$\left.\begin{aligned} \mathrm{D}&=\mathrm{X}x_{11}+\mathrm{Y}y_{11}+\mathrm{Z}z_{11}\\ \mathrm{D}'&=\mathrm{X}x_{12}+\mathrm{Y}y_{12}+\mathrm{Z}z_{12}\\ \mathrm{D}''&=\mathrm{X}x_{22}+\mathrm{Y}y_{22}+\mathrm{Z}z_{22}\end{aligned}\right\}, \qquad \ldots \quad (3)$$

X, Y, Z being $\dfrac{1}{\mathrm{V}}\begin{vmatrix} y_1 & z_1\\ y_2 & z_2\end{vmatrix}$, $\dfrac{1}{\mathrm{V}}\begin{vmatrix} z_1 & x_1\\ z_2 & x_2\end{vmatrix}$ and $\dfrac{1}{\mathrm{V}}\begin{vmatrix} x_1 & y_1\\ x_2 & y_2\end{vmatrix}$, where

$$\mathrm{V}=+\sqrt{\mathrm{EG}-\mathrm{F}^2}.$$

We now look upon the whole problem from the stand-point of differential equations.

From equations (2) we obtain by differentiation[1]

$$\left.\begin{aligned} &\Sigma x_1\cdot x_{11}=\tfrac{1}{2}\frac{\partial \mathrm{E}}{\partial u}, \quad \Sigma x_2\cdot x_{11}=\frac{\partial \mathrm{F}}{\partial u}-\tfrac{1}{2}\frac{\partial \mathrm{E}}{\partial v},\\ &\Sigma x_1\cdot x_{12}=\tfrac{1}{2}\frac{\partial \mathrm{E}}{\partial v}, \quad \Sigma x_2\cdot x_{12}=\tfrac{1}{2}\frac{\partial \mathrm{G}}{\partial u},\\ &\Sigma x_1\cdot x_{22}=\frac{\partial \mathrm{F}}{\partial v}-\tfrac{1}{2}\frac{\partial \mathrm{G}}{\partial u}, \quad \Sigma x_1\cdot x_{22}=\tfrac{1}{2}\frac{\partial \mathrm{G}}{\partial v}.\end{aligned}\right\} \qquad \ldots \quad (4)$$

Solving for x_{11}, x_{12}, x_{22} between equations (3) and (4),

$$\left.\begin{aligned} x_{11}&=\Gamma x_1+\Delta x_2+\mathrm{D}\mathrm{X},\\ x_{12}&=\Gamma' x_1+\Delta' x_2+\mathrm{D}'\mathrm{X},\\ x_{22}&=\Gamma'' x_1+\Delta'' x_2+\mathrm{D}''\mathrm{X}.\end{aligned}\right\} \qquad \ldots \quad (5)$$

[1] This is the standard method for the deduction of the Gaussian and the Codazzi relations. It has been repeated in outline for emphasising the analytical stand-point.

Using now the conditions of integrability of the type

$$\frac{\partial}{\partial v}(x_{11})=\frac{\partial}{\partial u}(x_{12}), \quad \ldots$$

we get the three fundamental equations first obtained by Gauss and Codazzi:

$$\frac{DD''-D'^2}{V^2}=\frac{1}{2V}\left\{\frac{\partial}{\partial u}\left[\frac{F}{EV}\frac{\partial E}{\partial v}-\frac{1}{V}\frac{\partial G}{\partial u}\right]\right.$$

$$\left.+\frac{\partial}{\partial v}\left[\frac{2}{V}\frac{\partial F}{\partial u}-\frac{1}{V}\frac{\partial E}{\partial v}-\frac{F}{EV}\frac{\partial E}{\partial u}\right]\right\}, \quad \ldots \quad (6)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial v}\left(\frac{D}{V}\right)-\frac{\partial}{\partial u}\left(\frac{D'}{V}\right)+\Delta''\frac{D}{V}-2\Delta'\frac{D'}{V}+\Delta\frac{D''}{V}=0,\\&\frac{\partial}{\partial u}\left(\frac{D''}{V}\right)-\frac{\partial}{\partial v}\left(\frac{D'}{V}\right)+\Gamma''\frac{D}{V}-2\Gamma'\frac{D'}{V}+\Gamma\frac{D''}{V}=0.\end{aligned}\right\} \quad \ldots \quad (7)$$

These equations in the new independent variables are therefore necessary.

It will be noticed that the Monge-Ampere equation of deformation is obtained by substituting the values of D, D', D'' obtained from (5) in the Gaussian Equation (6),

$$K=\frac{\{x_{11}-x_1\Gamma-x_2\Delta\}\{x_{22}-x_1\Gamma''-x_2\Delta''\}-\{x_{12}-x_1\Gamma'-x_2\Delta'\}^2}{X^2V^2} \ldots \quad (8)$$

and
$$\begin{aligned}X^2V^2&=(y_1z_2-y_2z_1)^2\\&=(y_1^2+z_1^2)(y_2^2+z_2^2)-(y_1y_2+z_1z_2)^2\\&=(E-x_1^2)(G-x_2^2)-(F-x_1x_2)^2\\&=V^2-[Gx_1^2-2Fx_1x_2+Ex_2^2).\end{aligned}$$

This agrees with the result given in Forsyth's Differential Geometry, p. 363.

Substituting for D, D′, D″ from equations (5) in equation 7(1), we have

$$\frac{\partial}{\partial v}\left(\frac{x_{11}-x_1\Gamma-x_2\Delta}{VX}\right)-\frac{\partial}{\partial u}\left(\frac{x_{12}-x_1\Gamma'-x_2\Delta'}{VX}\right)$$

$$+\Delta''\left(\frac{x_{11}-x_1\Gamma-x_2\Delta}{VX}\right)-2\Delta'\left(\frac{x_{12}-x_1\Gamma'-x_2\Delta'}{VX}\right)$$

$$+\Delta\left(\frac{x_{22}-x_1\Gamma''-x_2\Delta''}{VX}\right)=0, \quad \ldots \quad (9)$$

$$\text{or, } \frac{1}{VX}\Big[x_{112}-x_{12}\,\Gamma-x_{22}\Delta-\Gamma_2 x_1-x_2\Delta_2-x_{112}+x_{11}\,\Gamma'+x_{12}\Delta'$$

$$+x_1\Gamma'_1+x_2\Delta'_1+x_{11}\Delta''-2x_{12}\Delta'+x_{22}\cdot\Delta-\Delta''(x_1\Gamma+x_2\Delta)$$

$$+2\Delta'(x_1\Gamma'+x_2\Delta')-\Delta(x_1\Gamma''+x_2\Delta'')\Big]$$

$$-\frac{1}{V^2X^2}\Big[x_{11}-x_1\Gamma-x_2\Delta\Big]\frac{\partial(VX)}{\partial v}$$

$$+\frac{1}{V^2X^2}\Big[x_{12}-x_1\Gamma'-x_2\Delta'\Big]\frac{\partial(VX)}{\partial u}=0.$$

$$\text{Co eff. of } x_{11}VX=\Gamma'+\Delta''-\frac{\partial\log VX}{\partial v}$$

$$=\frac{\partial\log V}{\partial v}-\frac{\partial\log VX}{\partial v}=-\frac{\partial\log X}{\partial v}$$

[Eisenhart, Diff. Geom., Equation (3), Chapter V.]

$$\text{Co-eff. of } x_{12}\cdot VX=-\Gamma-\Delta'+\frac{\partial\log(VX)}{\partial u}$$

$$=-\frac{\partial\log V}{\partial u}+\frac{\partial\log VX}{\partial u}=-\frac{\partial\log X}{\partial u}.$$

Co-eff of x_1 $VX = -\Gamma_2 + \Gamma'_1 - \Delta''\Gamma + 2\Delta'\Gamma' - \Delta\Gamma''$

$$+\Gamma \frac{\partial \log VX}{\partial v} - \Gamma' \frac{\partial \log VX}{\partial u}$$

$$= -\Gamma_2 + \Gamma'_1 - \Delta''\Gamma + 2\Delta'\Gamma' - \Delta\Gamma''$$

$$+\Gamma \left\{ \frac{\partial \log X}{\partial v} + \Gamma' + \Delta'' \right\} - \Gamma' \left\{ \frac{\partial \log X}{\partial u} + \Gamma + \Delta' \right\}.$$

$$= -\Gamma_2 + \Gamma'_1 + \Delta'\Gamma' - \Delta\Gamma'' + \Gamma \frac{\partial \log X}{\partial v} - \Gamma \frac{\partial \log X}{\partial u}$$

$$= F \cdot K + \Gamma \frac{\partial \log X}{\partial v} - \Gamma' \frac{\partial \log X}{\partial u};$$

[Eisenhart, Equation (11), Chap. V]

using K to denote the right hand side expression of equation (6).

Co-eff. of x_2 $VX = -\Delta_2 + \Delta'_1 - \Delta\Delta'' + 2\Delta'^2 - \Delta\Delta''$

$$+\Delta \frac{\partial \log VX}{\partial v} - \Delta' \frac{\partial \log VX}{\partial u}$$

$$= -\Delta_2 + \Delta'_1 + 2\Delta\Delta'' + 2\Delta'^2 + \Delta \left\{ \frac{\partial \log X}{\partial v} + \Gamma' + \Delta'' \right\}$$

$$- \Delta' \left\{ \frac{\partial \log X}{\partial u} + \Gamma + \Delta' \right\}$$

$$= -\Delta_2 + \Delta'_1 - \Delta\Delta'' + \Delta'^2 + \Delta\Gamma' - \Delta'\Gamma + \Delta \frac{\partial \log X}{\partial v} - \Delta' \frac{\partial \log X}{\partial u}$$

$$= -EK + \Delta \frac{\partial \log X}{\partial v} - \Delta' \frac{\partial \log X}{\partial u}.$$

Equation (9) therefore becomes

$$-\frac{\partial \log X}{\partial v} \left[x_{11} - \Gamma x_1 - \Delta x_2 \right] + \frac{\partial \log X}{\partial u} \left[x_{12} - \Gamma' x_1 - \Delta' x_2 \right]$$

$$+ K(F x_1 - E x_2) = 0. \qquad \ldots \quad (10)$$

We have seen that

$$X^2=1-\frac{Ex_2{}^2-2Fx_1x_2+Gx_1{}^2}{V^2}=1-\Phi(x),$$

$\Phi(x)$ standing for Beltrami's first differential parameter.

Similarly from the second Codazzi equation we obtain by the same substitution

$$-\frac{\partial \log X}{\partial v}\left[x_{12}-\Gamma' x_1-\Delta' x_2\right]+\frac{\partial \log X}{\partial u}\left[x_{22}-\Gamma'' x_1-\Delta'' x_2\right]$$
$$+K(Gx_1-Fx_2)=0. \qquad \dots \quad (11)$$

The three equations (6), (7) in the variables D, D′, D″ or their equivalents (8), (10), (11) are therefore necessary.

To prove the sufficiency of the three equations (6), (7), we fall back on a theorem proved by Bonnet. He has shown that if six functions E, F, G, D, D′, D″ of two variables u, v are given which satisfy the Gauss and the Codazzi equations, a surface can be determined unique except as regards its position and orientation in space having

$$Edu^2+2Fdu\,dv+Gdv^2 \text{ and } Ddu^2+2D'du\,dv+D''dv^2$$

as the first and the second fundamental quadrated forms respectively and the determination of the surface requires the integration of a Riccati equation and quadratures.

[Eisenhart, Diff. Geom., p. 159.]

It is clear, therefore, that any solution of equations (6) and (7) will determine a surface deformable into the given surface.

On the stability of two co-axial rectilinear vortices of compressible fluid

BY

BIBHUTIBHUSAN DATTA

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The problem of the stability of the circular forms of two co-axial rectilinear vortices of incompressible fluid has been completely sloved by Poincare.[1] The object of the present paper is to extend his treatment to vortices of compressible fluid. The case, when the two vortices are outside each other, has been fully discussed by Dr. Chree[2] and by myself.

Taking the vortex to be parallel to the axis of z, the volocity components (u, v) at any point (x, y) in the fluid are given by[3]

$$u = -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \qquad \dots \quad (1)$$

$$v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x};$$

hence $$\nabla_1{}^2\phi = -\theta, \quad \nabla_1{}^2\psi = \zeta,$$

where $$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{and} \quad \nabla_1{}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Then by the theory of attraction,

$$\phi = -\frac{1}{2\pi}\iint \theta' \log r \, dx'dy',$$

$$\psi = \frac{1}{2\pi}\iint \zeta' \log r \, dx'dy', \qquad \dots \quad (2)$$

[1] "*Théorie des Tourbillons,*" Paris (1893) p. 165.

[2] "Vortices in a Compressible fluid." *Mess. Math.* Vol. XVII (1888), p. 105.

[3] Vide, Lamb's "*Hydrodynamics*" 4th Ed. (1916) p. 213.

θ' and ζ' being respectively the values of θ and ζ at the point (x', y') and

$$r=\{(x-x')^2+(y-y')^2\}^{\frac{1}{2}};$$

the double integrations include all portions of the plane xy for which θ and ζ do not vanish.

The equation of continuity is

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\theta=0.$$

Therefore $$\phi=\frac{1}{2\pi}\iint\frac{1}{\rho'}\frac{D\rho'}{Dt}\log r\,dx'dy'.$$

Since θ and ζ are functions of distance from the centre we have, by the application of the First Mean Value Theorem, for circular vortex of cross section σ

$$\phi=\frac{1}{2\pi}\cdot\frac{\sigma}{\rho_0}\frac{D\rho_0}{Dt}\log r, \qquad \ldots \quad (3)$$

$$\psi=\frac{\zeta_0\sigma}{2\pi}\log r$$

where ρ_0 and ζ_0 are the mean values of the density and vorticity defined by the relations

$$\rho_0\sigma=\iint\rho'\,dx'dy', \quad \zeta_0\sigma=\iint\zeta'\,dx'dy'.$$

Again the equation of continuity is

$$\frac{D\rho_0\sigma}{Dt}=0$$

or, $$\frac{1}{\rho_0}\frac{D\rho_0}{Dt}+\frac{1}{\sigma}\frac{D\sigma}{Dt}=0.$$

Since σ always refers to a definite column of fluid, there is no difference between $\frac{D}{Dt}$ and the partial differential co-efficient $\frac{d}{dt}$.

Thus for a truly circular vortex of radius a and cross section σ, we have,

when $r>a$,

$$\phi_0=-\frac{1}{2\pi}\frac{d\sigma}{dt}\log r, \qquad \psi_0=\frac{k}{2\pi}\log r\,;$$

and when $r<a$,

$$\phi_0=-\frac{1}{4\pi a^2}\frac{d\sigma}{dt}r^2, \qquad \psi_0=\frac{k}{4\pi a^2}r^2,$$

where k is the strength of the vortex.

Consider two coaxial cylindrical vortices C and C′, of radius a and b and of vorticity ζ and ζ' respectively. Suppose C to be inside C′. Now let the cross sections of the two tubes be slightly deformed so that at any instant their boundaries are given by

$$r=a+\Sigma(a_n \cos n\theta+\beta_n \sin n\theta), \qquad \dots \quad (4)$$

$$r=b+\Sigma(a'_n \cos n\theta+\beta'_n \sin n\theta). \qquad \dots \quad (5)$$

Then we shall have, from the continuity of ϕ and ψ, and of radial and tangential velocities at the surface, for C :

when $r<a$,

$$\psi_c=\text{const}+\frac{k}{4\pi a^2}r^2-\Sigma\ \frac{k}{2\pi na}(a_n\cos n\theta+\beta_n \sin n\theta)\left(\frac{r}{a}\right)^n;$$

$$\phi_c=\text{const}-\frac{1}{4\pi a^2}\frac{d\sigma}{dt}r^2+\Sigma\ \frac{1}{2\pi na}\frac{d\sigma}{dt}(a_n \cos n\theta+\beta_n \sin n\theta)\left(\frac{r}{a}\right)^n,$$

when $r>a$,

$$\psi'_c=\text{const}+\frac{k}{2\pi}\log r-\Sigma\ \frac{k}{2\pi na}(a_n \cos n\theta+\beta_n \sin n\theta)\left(\frac{a}{r}\right)^n;$$

$$\phi'_c=\text{const}-\frac{1}{2\pi}\frac{d\sigma}{dt}\log r+\Sigma\ \frac{1}{2\pi na}\frac{d\sigma}{dt}(a_n \cos n\theta+\beta_n \sin n\theta)\left(\frac{a}{r}\right)^n,$$

and for C′ :

when $r<b$,

$$\psi_c'=\text{const}+\frac{k'}{4\pi b^2}r^2-\Sigma\frac{k'}{2\pi nb}(\alpha'_n\ \cos n\theta+\beta'_n\ \sin n\theta)\left(\frac{r}{b}\right)^n$$

$$\phi_c'=\text{const}-\frac{1}{4\pi b^2}\frac{d\sigma'}{dt}r^2+\Sigma\frac{1}{2\pi nb}\frac{d\sigma'}{dt}(\alpha'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{r}{b}\right)^n,$$

when $r>b$,

$$\psi'_c{}'=\text{const}+\frac{k'}{2\pi}\log r-\Sigma\frac{k'}{2\pi nb}(\alpha'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{b}{r}\right)^n,$$

$$\phi'_c{}'=\text{const}-\frac{1}{2\pi}\frac{d\sigma'}{dt}\log r+\Sigma\frac{1}{2\pi nb}\frac{d\sigma'}{dt}(\alpha'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{b}{r}\right)^n.$$

If $\mathfrak{R}$ denote the radial velocity of a point on the cylinder C and $a\Theta$ the velocity perpendicular to the radius vector, we have from (4),

$$\mathfrak{R}=\frac{da}{dt}+\Sigma\left(\frac{d\alpha_n}{dt}\cos n\theta+\frac{d\beta_n}{dt}\sin n\theta\right)-\Sigma n\Theta(\alpha_n\sin n\theta-\beta_n\cos n\theta). \tag{6}$$

Since powers and products of the small quantities α_n, β_n are to be neglected we may take in (6),

$$\Theta=\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2} \quad \ldots \tag{7}$$

We have also

$$\mathfrak{R}=-\frac{\partial\psi'_c}{r\partial\theta}-\frac{\partial\phi'_c}{\partial r}-\frac{\partial\psi_c'}{r\partial\theta}-\frac{\partial\phi_c}{\partial r}-\frac{a}{2\pi b^2}\frac{d\sigma'}{dt}; \quad \ldots \tag{8}$$

where r is to be put equal to $a+\Sigma(\alpha_n\cos n\theta+\beta_n\sin n\theta)$ after differentiation.

The two values of $\mathfrak{R}$ given by (6) and (8) must be identical. The terms independent of θ simply verify the known result $\frac{d\sigma}{dt}=2\pi a$

$\times \frac{da}{dt}$. Equating the co-efficients of $\cos n\theta$ and $\sin n\theta$, we get

$$\left.\begin{aligned}\frac{d\alpha_n}{dt} &= -n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right)\beta_n+\frac{k}{2\pi a^2}\beta_n+\frac{k'}{2\pi b^2}\epsilon^{n-1}\beta'_n\\ &\qquad +\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}\alpha_n-\frac{\epsilon^{n-1}}{2\pi b^2}\frac{d\sigma'}{dt}\alpha'_n,\\ \frac{d\beta_n}{dt} &= n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right)\alpha_n-\frac{k}{2\pi a^2}\alpha_n-\frac{k'}{2\pi b^2}\epsilon^{n-1}\alpha'_n\\ &\qquad +\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}\beta_n-\frac{\epsilon^{n-1}}{2\pi b^2}\frac{d\sigma'}{dt}\beta'_n;\end{aligned}\right\} \quad (9)$$

where $\epsilon = a/b$. Similarly from the condition to be satisfied at the surface of the vortex C′ we get the equations:

$$\left.\begin{aligned}\frac{d\alpha'_n}{dt} &= -n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right)\beta'_n+\frac{k}{2\pi a^2}\epsilon^{n+1}\beta_n+\frac{k'}{2\pi b^2}\beta'_n\\ &\qquad -\frac{1}{2\pi b^2}\frac{d\sigma}{dt}\alpha'_n+\frac{\epsilon^{n+1}}{2\pi a^2}\frac{d\sigma}{dt}\alpha_n,\\ \frac{d\beta'_n}{dt} &= n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right)\alpha'_n-\frac{k}{2\pi a^2}\epsilon^{n+1}\alpha_n-\frac{k'}{2\pi b^2}\alpha'_n\\ &\qquad -\frac{1}{2\pi b^2}\frac{d\sigma}{dt}\beta'_n+\frac{\epsilon^{n+1}}{2\pi a^2}\frac{d\sigma}{dt}\beta_n.\end{aligned}\right\} \quad (10)$$

For abbreviation, put

$$\left.\begin{aligned} f&=\frac{k}{2\pi a^2}-n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right), \quad g=\frac{k'}{2\pi b^2}\epsilon^{n-1}, \quad h=\frac{1}{2\pi b^2}\frac{d\sigma}{dt};\\ f'&=\frac{k'}{2\pi b^2}-n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right); \quad g'=\frac{k}{2\pi a^2}\epsilon^{n+1}, \quad h'=\frac{1}{2\pi b^2}\frac{d\sigma'}{dt},\end{aligned}\right\} \quad (11)$$

The equations (9) and (10) can, then, be written as

$$\left.\begin{aligned} \frac{d\alpha_n}{dt} &= f\beta_n + g\beta'_n + h'(\alpha_n - \epsilon^{n-1}\alpha'_n), \\ \frac{d\beta_n}{dt} &= -f\alpha_n - g\alpha'_n + h'(\beta_n - \epsilon^{n-1}\beta'_n), \\ \frac{d\alpha'_n}{dt} &= f'\beta'_n + g'\beta_n + h(\epsilon^{n-1}\alpha_n - \alpha'_n), \\ \frac{d\beta'_n}{dt} &= -f'\alpha'_n - g'\alpha_n + h(\epsilon^{n-1}\beta_n - \beta'_n). \end{aligned}\right\} \quad \dots \quad (12)$$

Denote

$$\left.\begin{aligned} u &= \alpha_n + i\beta_n, \\ v &= \alpha_n - i\beta_n\ ; \end{aligned}\right\} \quad \left.\begin{aligned} u' &= \alpha'_n + i\beta'_n, \\ v' &= \alpha'_n - i\beta'_n\ ; \end{aligned}\right\} \quad \dots \quad (13)$$

where $i = \sqrt{-1}$.

Then from equations (12), we get

$$\begin{aligned} \frac{du}{dt} - (h' - if)u &= -(h'\epsilon^{n-1} + ig)u', \\ \frac{dv}{dt} - (h' + if)v &= -(h'\epsilon^{n-1} - ig)v', \\ \frac{du'}{dt} + (h + if')u' &= (h\epsilon^{n-1} - ig')u, \\ \frac{dv'}{dt} + (h - if')v' &= (h\epsilon^{n-1} + ig')v. \end{aligned} \quad \dots \quad (14)$$

Eliminating u' between the first and the third of these equations we get

$$\frac{d}{dt}\left[\frac{e^{\int(h+if')dt}}{h'\epsilon^{n-1}+ig}\left\{\frac{du}{dt} - (h' - if)u\right\}\right] + (h\epsilon^{n-1} - ig')ue^{\int(h+if')dt} = 0\ ;$$

and similarly we obtain the differential equations for u', v and v' as

$$\frac{d}{dt}\left[\frac{e^{\int(h-if')dt}}{h'\epsilon^{n-1}-ig}\left\{\frac{dv}{dt}-(h'+if)v\right\}\right]+(h\epsilon^{n-1}+ig')ve^{\int(h-if')dt}=0,$$

$$\frac{d}{dt}\left[\frac{e^{-\int(h'-if)\,dt}}{h\epsilon^{n-1}-ig'}\left\{\frac{du'}{dt}+(h+if')\,u'\right\}\right]$$

$$+(h'\epsilon^{n-1}+ig)u'e^{-\int(h'-if)dt}=0,$$

$$\frac{d}{dt}\left[\frac{e^{-\int(h'+if)dt}}{h\epsilon^{n-1}+ig'}\left\{\frac{dv'}{dt}+(h-if')v'\right\}\right]$$

$$+(h'\epsilon^{n-1}-ig)v'e^{-\int(h'+if)dt}=0.$$

Simplifying we get the differential equations for the determination of u, u', v, v' as

$$\left.\begin{aligned}&\frac{d^2u}{dt^2}+P\frac{du}{dt}+Qu=0,\\&\frac{d^2v}{dt^2}+R\frac{dv}{dt}+Sv=0,\\&\frac{d^2u'}{dt^2}+P'\frac{du'}{dt}\;Q'u'=0,\\&\frac{d^2v'}{dt^2}+R'\frac{dv'}{dt}+S'v'=0\,;\end{aligned}\right\}\qquad(15)$$

where

$$P=(h+if')-(h'-if)-\frac{d}{dt}(h'\epsilon^{n-1}+ig)\,/(h'\epsilon^{n-1}+ig)\,,$$

$$P'=(h+if')-(h'-if)\;-\frac{d}{dt}(h\epsilon^{n-1}-ig')/(h\epsilon^{n-1}-ig')\,,$$

$$R=(h-if')-(h'+if)-\frac{d}{dt}(h'\epsilon^{n-1}-ig)/(h'\epsilon^{n-1}-ig)\,,$$

$$R'=(h-if')-(h'+if)-\frac{d}{dt}(h'\epsilon^{n-1}+ig)/(h'\epsilon^{n-1}+ig)\,,$$

$$Q=(h\epsilon^{n-1}-ig')(h'\epsilon^{n-1}+ig)-\frac{d}{dt}(h'-if)$$

$$-(h'-if)\{(h+if')-\frac{d}{dt}(h'\epsilon^{n-1}+ig)/(h'\epsilon^{n-1}+ig)\},$$

$$Q'=(h'\epsilon^{n-1}+ig)(h\epsilon^{n-1}-ig')+\frac{d}{dt}(h+if')$$

$$-(h+if')\{(h'-if)+\frac{d}{dt}(h\epsilon^{n-1}-ig')/(h\epsilon^{n-1}-ig')\},$$

$$S=(h'\epsilon^{n-1}-ig')(h'\epsilon^{n-1}+ig)-\frac{d}{dt}(h'+if)$$

$$-(h'+if)\{(h-if')-\frac{d}{dt}(h'\epsilon^{n-1}-ig/(h'\epsilon^{n-1}-ig)\}.$$

$$S'=(h'\epsilon^{n-1}-ig)(h\epsilon^{n-1}+ig)+\frac{d}{dt}(h'-if')$$

$$-(h-if')\{(h'+if)+\frac{d}{dt}(h\epsilon^{n-1}-ig')/(h\epsilon^{n-1}-ig')\}; \qquad (16)$$

These are equations of second degree with variable co-efficients and at the present state of mathematical analysis they cannot be generally solved. But that will not deter us from the discussion of our problem. For the necessary and sufficient condition for the stability of the circular from is that α's and β's, therefore u, u'...must be expressible in the form[1]

$$e^{itp}$$

where p is a real quantity.

Now take the first of the equations (15), *viz.*

$$\frac{d^2u}{dt^2}+P\frac{du}{dt}+Qu=0.$$

This can be transformed into the form

$$\frac{d^2v}{dt^2}+Iv=0 \qquad \ldots \quad (17)$$

[1] Poincaré, *l.c.* p. 169.

where
$$u = ve^{-\frac{1}{2}\int P dt}, \qquad \dots \quad (18)$$

and
$$I = Q - \tfrac{1}{2}\frac{dP}{dt} - \tfrac{1}{4}P^2. \qquad \dots \quad (19)$$

Now P is a complex quantity of the from $\beta + i\gamma$; hence we conclude that the motion is unstable.

Put $a = a_n = \beta_n = 0$; so that there is only a single compressible vortex of radius b. then, from (11)

$$f = -\frac{k'}{2\pi b^2};\; f' = (1-n)\frac{k'}{2\pi b^2},\; h' = \frac{1}{2\pi b^2}\frac{d\sigma'}{dt}: \qquad \dots \quad (20)$$

$$\epsilon = g = h = g' = 0.$$

The equations (14) then reduce to

$$\frac{du'}{dt} - i(n-1)\frac{k'}{2\pi b^2}u' = 0,$$

$$\frac{dv'}{dt} + i(n-1)\frac{k'}{2\pi b^2}v' = 0;$$

whence

$$u' = A \exp.\left\{ i(n-1)\int_0^t \frac{k'}{2\pi b^2}dt \right\}, \qquad \dots \quad (21)$$

$$v' = A \exp.\left\{ -i(n-1)\int_0^t \frac{k'}{2\pi b^2}dt \right\},$$

where A is a constant of integration.

Therefore

$$a'_n = A\cos\left\{ k'(n-1)\int_0^t \frac{dt}{\sigma'} \right\}, \qquad \dots \quad (22)$$

$$\beta'_n = A\sin\left\{ k'(n-1)\int_0^t \frac{dt}{\sigma'} \right\},$$

which agree with the results of Dr. Chree.[1]

[1] *l.c.*, p. 112.

Again let $h=h'=0$; so that there are two co-axial vortices of incompressible fluid. In this case f, f', g, g'...are independent of time; then

$$P=P'=i(f+f'); \qquad R=R'=-i(f+f'), \qquad \dots \quad (23)$$

$$Q=Q'=gg'-ff', \qquad S=S'=ff'-gg'.$$

The first equation of (15) becomes

$$\frac{d^2u}{dt^2}+i(f+f')\frac{du}{dt}+(gg'-ff')u=0. \qquad \dots \quad (24)$$

Assume $u=e^{iat}$.

Substituting in the previous equation, we get the equation for a as

$$a^2+(f+f')a-(gg'-ff')=0. \qquad \dots \quad (25)$$

The necessary and sufficient condition of stability is that a must be real; therefore

$$(f+f')^2+4(gg'-ff')>0,$$

or

$$(f-f')^2+4gg'>0,$$

substituting the values of f, f', g, g' from (11), get

$$[\zeta-n(\zeta+\zeta')-\zeta'+n(\zeta\epsilon^2+\zeta')]^2+4\zeta\zeta'\epsilon^{2n}>0 \qquad \dots \quad (26)$$

which is the condition obtained by Poincaré.[1]

[1] *l.c.*, p. 172.

On some Properties of Natural Numbers

BY

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Here in this note a few theorems in natural numbers have been proved. The truth of some of these theorems may be perceived by merely observing the number when written in order of magnitude.

Let N, M and P be three positive integers and $N=M+P$. And let us express them in the scale of another positive integer r.

$$N=a_n r^n+a_{n-1}r^{n-1}+\dots+a_1 r+a_0,$$

$$M=b_n r^n+\dots+b_1 r+b_0,$$

$$P=c_n r^n+\dots+c_1 r+c_0,$$

and

$$\phi(N)=na_n r^n+(n-1)a_{n-1}r^{n-1}+\dots+a_1 r.$$

The co-efficients a, b, c are all less than r.

I. $$\Sigma a \not> \Sigma b+\Sigma c,$$

i.e., $$\not> \Sigma(b+c).$$

If $\Sigma a>\Sigma(b+c)$, then at least one of the a's, say a_s, will be greater than (b_s+c_s) and the first $(s-1)$ a's are equal to the corresponding $(b+c)$'s.

$$a_n=b_n+c_n \;,\dots,\; a_{s-1}=b_{s-1}+c_{s-1},$$

$$a_s\neq b_s+c_s.$$

We have

$$a_n r^n+\dots+a_s r^s+\dots+a_0$$

$$=(b_n+c_n)r^n+\dots+(b_s+c_s)r^s+\dots+(b_0+c_0).$$

Now by re-arranging the terms on the right of $(b_s+c_s)r^s$, we obtain not only all the terms on the right of $a_s r^s$ but also $(a_s-b_s-c_s)\,r^s$; for $(b_s+c_s)\,r^s+(a_s-b_s-c_s)\,r^s=a_s r^s$. $(a_s-b_s-c_s)\,r^s$ has been obtained as the sum of parts of some or all the terms following $a_s r^s$. This sum is the sum of parts of co-efficients multiplied by quantities less than r^s. Hence this sum of the co-efficients is greater than. $(a_s-b_s-c_s)$. Therefore

$$\Sigma a<\Sigma(b+c).$$

From this it follows that

(i) $a_s = b_s + c_s + 1$,

(ii) if any of the a's is not equal to the corresponding $(b+c)$ then $\Sigma a < \Sigma(b+c)$; and *vice-versa*,

(iii) if $\Sigma a = \Sigma(b+c)$ then the a's are separately equal to the corresponding $(b+c)$'s.

II. $$\phi(N) \nless \phi(M)+\phi(P).$$

Suppose $$a_n = b_n + c_n ; \ldots, a_{s+1} = b_{s+1} + c_{s+1},$$

and $$a_s \neq b_s + c_s.$$

As a_s is greater than (b_s+c_s) by unity, so there must be some co-efficient of $(M+P)$, say (b_p+c_p), which will be greater than a_p by r; that is, $a_p + r = b_p + c_p$, and the co-efficients of $(M+P)$ between (b_s+c_s) and (b_p+c_p) must be

$$(a_{p-1}+r-1),\ (a_{s-2}+r-1), \ldots, (a_{p+1}+r-1).$$

If $$\phi(N) = A(r) + sa_s r^s + B(r),$$

then $$\phi(M)+\phi(P) = A(r) + s(a_s-1)r^s + B(r)$$

$$+\left\{(s-1)r^s - r\frac{r^{s-1}-1}{r-1}\right\} - \left\{(p-1)r^p - r\frac{r^{p-1}-1}{r-1}\right\} + pr^p.$$

Now $$\phi(N) > \phi(M)+\phi(P),$$

if $$sa_s r^s > \left(sa_s - 1 - \frac{1}{r-1}\right)r^s + \left(1+\frac{1}{r-1}\right)r^p, \qquad \ldots \quad (\alpha)$$

i.e., if $$\left(1+\frac{1}{r-1}\right)r^s > \left(1+\frac{1}{r-1}\right)r^p.$$

(i) As $s>p$, hence $\phi(N) > \phi(M)+\phi(P)$.

(ii) *If all the a's are separately equal to the corresponding* $(b+c)$'s, *then*

$$\phi(N) = \phi(M)+\phi(P).$$

(iii) From (α) we see that the difference

$$\phi(N) - \{\phi(M)+\phi(P)\}$$

$$= \left(1+\frac{1}{r-1}\right)r^s - \left(1+\frac{1}{r-1}\right)r^p$$

which is a positive quantity.

As s increases and p decreases, the difference increases. The larger or the more numerous are the gaps the greater is the difference. That is to say as $\Sigma(b+c)$ increases the value of $\{\phi(M)+\phi(P)\}$ decreases. The greatest value of $\Sigma(b+c)$

$$=(a_n-1)+(a_{n-1}+r-1)+\ldots+(a_2+r-1)+(a_1+r)$$

$$=\Sigma a+n(r-1).$$

Therefore *the least value of* $\{\phi(M)+\phi(P)\}$

$$=\phi(N)-\left(1+\frac{1}{r-1}\right)r^n+\frac{r}{r-1}.$$

III. The factors of natural numbers, when written in order of magnitude, which are of the form r^s (where $s=1,2,\ldots$), are

$$\left.\begin{array}{l} r,\ r,\ \ldots\ r,\ r^2 \\ r,\ r,\ \ldots\ r,\ r^2 \\ \ldots\ \ \ldots\ \ \ldots \\ r,\ r,\ \ldots\ r,\ r^3 \\ r,\ r,\ \ldots\ r,\ r^2 \\ r,\ r,\ \ldots\ r,\ r^2 \\ \ldots\ \ \ldots\ \ \ldots \end{array}\right\} \qquad \ldots (\alpha)$$

The first row contains the first r—factors, the second row contains the second r—factors and so on. The first $(r-1)$ factors of all the rows are r. The last factors of the first $(r-1)$ rows are r^2. The last factor of the rth row is r^3. Of the (r^2-1) rows the last factor, of every rth row, is r^3, and the last factors of all the other rows are r^2. The last factor of the r^2th row is r^4.

The sum of the powers of r up to the first r is $\frac{r-1}{r-1}$,

... ... r^2 is $\frac{r^2-1}{r-1}$,

... ... r^n is $\frac{r^n-1}{r-1}$.

(i) *In the product of the first N factors, the powers of* r

$$=a_n\frac{r^{n+1}-1}{r-1}+a_{n-1}\frac{r^n-1}{r-1}+\ldots+a_0\frac{r-1}{r-1}$$

$$=\frac{r}{r-1}\,\mathrm{N}-\frac{1}{r-1}\,\Sigma a.$$

In the joint product of the first M and the first P *factors the powers of* r

$$=\frac{r}{r-1}\,(\mathrm{M}+\mathrm{P})-\frac{1}{r-1}\,\Sigma(b+c).$$

As $\Sigma a \lesseqgtr \Sigma(b+c)$, therefore ,

(ii) *the product of the first M factors is either equal to or less than the product of any other M consecutive factors.*

(iii) if r be a prime number, then at once follows the known theorem, namely, *the product of M consecutive numbers is divisible by M !* That is, $\binom{x}{M}$ is an integer, where x is the first term of the M consecutive numbers.

(iv) *If the highest power of* r *in the M consecutive factors be greater than the highest power of* r *in the first M factors, then the product of the M consecutive factors will be greater than the product of the first M factors.*

Let $\qquad x=u_sr^s+u_{s-1}r^{s-1}+\ldots+u_1r+u_0\,.$

and $\qquad x+\mathrm{M}=v_mr^m+\ldots+v_1r+v_0$

Suppose $\qquad u_sr^s=v_mr^m,\ldots u_{s-w-1}r^{s-w-1}=v_{s-w-1}r^{m-w-1},$

and $\qquad u_{s-w}r^{s-w}\neq v_{s-w}r^{m-w},$

$$v_{s-w}r^{m-w}\gtrless b_nr^n.$$

If $v_{s-w}r^{m-w}>b_nr^n$, *then the integer* $\binom{x}{M}$ *is divisible by* r.

(v) *If* $\mathrm{M}+\mathrm{P}=\mathrm{N}-1$, *then*

$$\frac{r}{r-1}\,\mathrm{N}-\frac{1}{r-1}\,\Sigma a<\frac{r}{r-1}\,(\mathrm{M}+\mathrm{P})-\frac{1}{r-1}\,\Sigma(b+c)+(n+1).$$

Therefore $\qquad \Sigma(b+c)-\Sigma a<(r-1)(n+1).$

(vi) More generally, *to find the product of the factors of the form* $r_1^{x_1} r_2^{x_2} \ldots r_a^{x_a}$ *which is contained in n !, where the r's are prime numbers,*

Let the product be $\left(r_1^{x_1+1} r_2^{x_2+1} \ldots\right)$ and the integral part of $\frac{r}{r_1 r_2 \ldots r_a}$ be m.

Let
$$m = a_1 r_1^{s_1} + a_2 r_1^{s_2} + \ldots + a_u r_1^{s_u}$$
$$= b_1 r_2^{t_1} + b_2 r_2^{t_2} + \ldots + b_v r_2^{t_v}$$
$$= c_1 r_3^{k_1} + c_2 r_3^{k_2} + \ldots + c_w r^{k_w}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Hence
$$n_1 = \frac{r_1}{r_1 - 1} m - \frac{1}{r_1 - 1} (a_1 + a_2 + \ldots + a_u)$$
$$n_2 = \frac{r_2}{r_2 - 1} m - \frac{1}{r_2 - 1} (b_1 + b_2 + \ldots + b_v)$$
$$n_3 = \frac{r_3}{r_3 - 1} m - \frac{1}{r_3 - 1} (c_1 + c_2 + \ldots + c_w)$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

The sum of these factors
$$= \frac{m(m+1)}{2} r_1 r_2 \ldots r_x.$$

IV. From (a) III, we have

The sum of the factors up to the first $r = r$,

„ „ $r^2 = r^2 + (r-1)r$,

„ „ $r^3 = r^3 + 2(r-1)r^2$.

... ...

„ „ $r^n = r^n + (n-1)(r-1)r^{n-1}$.

(i) *The sum of the first* N *factors of* (a) *III.*
$$= rN + (r-1)\phi(N).$$

The sum of the first M *factors*

$$=rM+(r-1)\phi(M).$$

The sum of the first P *factors*

$$=rP+(r-1)\phi(P).$$

(ii) As $\phi(N) \geqq \phi(M)+\phi(P)$,

hence *the sum of any M consecutive factors is either equal to or greater than the sum of the first M factors.*

V. *To find the product of all the numbers which are of the form* $r_1^{p_1} r_2^{p_2}$ *and less than or equal to a given number N. Neither* p_1, *nor* p_2 *is zero and both* r_1 *and* r_2 *are primes.*

Let $r_1 < r_2$; and the integral part of $\dfrac{N}{r_1}$

$$=b_1 r_2^{s}+b_2 r_2^{s-1}+\ldots+b_s r_2+b_{s+1}$$

And

$$b_1=c_1 r_1^{x_1-1}+\ldots,$$

$$b_1 r_2+b_2=c_2 r_1^{x_2-1}+\ldots,$$

$$b_1 r_2^{2}+b_2 r+b_3=c_3 r_1^{x_3-1}+\ldots,$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$b_1 r_2^{s-1}+\ldots+b_s=c_s r_1^{x_s-1}+\ldots.$$

The highest number which contains r_2^{s} *is* $r_2^{s} r_1^{x_1}$

for
$$r_2^{s} r_1^{x_1} < r_2^{s}(c_1 r_1^{x_1-1}+\ldots)r_1$$
$$< b_1 r_2^{s} r_1$$
$$< n ;$$

but
$$r_2^{s} r_1^{x_1+1} \geqq r_1\left\{(b_1+1)r_2^{s}+\ldots\right\}$$
$$> r_1 \frac{n}{r_1}$$
$$> n.$$

The highest number which contains r_2^{s-1} is $r_2^{s-1} r_1^{x_2}$

for $$r_2^{s-1} r_1^{x_2} < \left\{ c_2 r_1^{x_1-1} + \dots \right\} r_2^{s-1}$$

$$< r_1 (b_1 r_1 + b_2) r_2^{s-1}$$

$$< n\,;$$

but $$r_2^{s-1} r_1^{x_2+1} \geq r_1 r_2^{s-1} (b_1 r_1 + b_2 + 1)$$

$$> r_1 \frac{n}{r_1}$$

$$> n.$$

In this way we can show that the product

$$= \left\{ r_2^{x_1 s} \times r_1^{\frac{1}{2} x_1 (x_1+1)} \right\} \left\{ r_2^{x_2 (s-1)} r_1^{\frac{1}{2} x_2 (x_2+1)} \right\}$$

$$\dots \left\{ r_2^{x_s} r_1^{\frac{1}{2} x_s (x_s+1)} \right\}.$$

And the number of these numbers $=(x_1 + x_2 + \dots + x_s)$.

And their sum

$$= r_2^{s} \frac{r_1^{x_1+1} - 1}{r_1 - 1} + r_2^{(s-1)} \frac{r_1^{x_2+1} - 1}{r_1 - 1} + \dots + r_2 \frac{r_1^{x_s+1} - 1}{r - 1}.$$

For practical purposes, in finding the values of x's, the following method may be adopted with advantage over the other.

If $$r_2 = b_1 r_1^{p} + \dots,$$

then the highest term containing

$$r_2^{s} = r_2^{s} r_1^{x_1+1} \text{ and } r_2^{s-1} \not< r_2^{s-1} r_1^{x_1+p+1}$$

for $$r_2^{s-1} r_1^{x_1+p+1} < r_2^{s} r_1^{x_1+1}$$

Similarly the highest term containing

$$r_2^{s-2} \not< r_2^{s-2} r_1^{x_1+2p+1} \quad \text{and} \quad r_2^{s-3} \not< r_2^{s-3} r_1^{x_1+3p+1}$$

and so on.

Now multiplying

$$r_2^{s-1} r_1^{x_1+p+1} \text{ by } r_1^{D} \quad (D=0,1,\ldots),$$

we can find x_2. In this way we can find the values of other x's.

VI. *To find the product of all numbers of the form* $r_1^{p_1} r_2^{p_2} r_3^{p_3}$ *and less than or equal to a given number N,* where r_1, r_2, and r_3 are prime numbers and none of the p's are zero.

Let the integral part of $\dfrac{N}{r_1 r_2 r_3}$ be M_1. The product of all the numbers of the form $r_1^{p_1} r_2^{p_2}$, $r_1^{p_1} r_3^{p_3}$ and $r_2^{p_2} r_3^{p_3}$ and not greater than M_1, are respectively

$$\left\{ r_2^{s x_1} r_1^{\frac{1}{2}x_1(x_1+1)} \right\} \cdots \left\{ r_2^{x_s} r_1^{\frac{1}{2}x_s(x_s+1)} \right\} \qquad \ldots \quad (1)$$

$$\left\{ r_3^{t y_1} r_1^{\frac{1}{2}y_1(y_1+1)} \right\} \cdots \left\{ r_3^{y_t} r_1^{\frac{1}{2}y_t(y_t+1)} \right\} \qquad \ldots \quad (2)$$

and

$$\left\{ r_3^{u z_1} r_2^{\frac{1}{2}z_1(z_1+1)} \right\} \cdots \left\{ r_3^{z_u} r_2^{\frac{1}{2}z_u(z_u+1)} \right\}. \qquad \ldots \quad (3)$$

Let

$$M_1 = \alpha r_1^{\theta} + \ldots$$

$$= \beta r_2^{\sigma} + \ldots$$

$$= \gamma r_3^{\delta} + \ldots.$$

Thus we get the numbers

$$r_1 r_2 r_3\, r_1^{\frac{\theta}{2}(\theta+1)} \qquad \ldots \quad (4)$$

$$r_1 r_2 r_3\, r_2^{\frac{\sigma}{2}(\sigma+1)}, \qquad \ldots \quad (5)$$

and

$$r_1 r_2 r_3\, r_3^{\frac{\delta}{2}(\delta+1)} \qquad \ldots \quad (6)$$

Let us denote the combined product of (1), (2), (3), (4), (5) and (6) by $r_1 r_2 r_3 \psi(M_1)$.

Of these M_1 numbers, there are numbers which contain $r_1 r_2 r_3$ as factors.

Let the integral part of $\dfrac{M_1}{r_1 r_2 r_3}$ be M_2. Here we obtain the product $(r_1 r_2 r_3)^2 \psi(M_2)$. Continuing this process we shall come to a number M_{m+1} which will be less than $r_1 r_2 r_3$.

Thus the required product

$$= r_1 r_2 r_3 \psi(M_1)(r_1 r_2 r_3)^2 \psi(M_2) \ldots$$

$$\ldots (r_1 r_2 r_3)^m \psi(M_m)(r_1 r_2 r_3)^{1+2+\ldots+m}$$

$$= \psi(M_1)\psi(M_2)\ldots\psi(M_m)(r_1 r_2 r_3)^{\frac{m(m+1)}{2}}$$

As for example let us find the product of all the numbers of the form $2^{p_1} . 3^{p_2} . 5^{p_3}$ and less than or equal to 9000.

Here

$$\frac{9000}{2.3.5} = 300,$$

$$\frac{300}{3} = 100 = 2^6 + \ldots,$$

$$\frac{300}{2} = 150 = 2^4 + \ldots$$

Hence the numbers of the form $2^{p_1}.3^{p_2}.2.3.5$ and not greater than 9000, are

(1) $3^4 2.2.3.5$; $3^3\{2.2^2.2^3\}$*$2.3.5$; $3^2\{2.2^2.2^3.2^4.2^5\}2.3.5$

and $3\{2.2^2.2^3.2^4.2^5.2^6\}2.3.5$.

Those which are of the form $2^{p_1}.5^{p_3}.2.3.5$ are

(2) $5^3.2.2.3.5$; $5^2\{2.2^2.2^3\}2.3.5$; $5\{2.2^2.2^3.2^4.2^6\}2.3.5$.

Those of the form $3^{p_2}.5^{p_3}.2.3.5$ are

(3) $5^2\{3.3^2\}2.3.5$; $5\{3.3^2.3^3\}2.3.5$.

The numbers of the form $\left\{2^{p_1}, 3^{p_2}, 5^{p_3}\right\}2.3.5$ respectively are

(4) $\{2.2^2.2^3 2^4.2^5.2^6\}2.3.5$,

(5) $\{3.3^2.3^3.3^4\}2.3.5$,

(6) $\{5.5^2.5^3\}2.3.5$.

Again

$$\frac{300}{2.3.5}=10,\ \frac{10}{3}=2+1\ ;\ \frac{10}{2}=3+2.$$

Hence in addition to the above we have also the following numbers

(7) $3.2.(2.3.5)^2$

(8) $5.2.(2.3.5)^2$

(9) $\{2.2^2.2^3\}(2.3.5)^2$

(10) $\{3.3^2\}(2.3.5)^2$

(11) $5.(2.3.5)^2$

(12) $2.3.5$; $(2.3.5)^2$.

The number of these numbers is 52; the product of these 52 numbers is $2^{149}.3^{107}.5^{84}$.

* $[a\{x.y.z\}b]=abx.aby.abz$.

A Note on Whittaker's Formula for the Solution of Algebraic or Transcendental Equations

BY

BHOLANATH PAL

In a recent paper, published in the Proceedings of the Edinburgh Mathematical Society (November, 1918), Prof. Whittaker has communicated the following elegant formula for obtaining the root, which is the smallest in absolute value, of algebraic or transcendental equations. He has shown that the root of the equation

$$0=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots$$

which is the smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2\,a_2}{a_1\begin{vmatrix}a_1 & a_2\\ a_0 & a_1\end{vmatrix}}-\frac{a_0^3\begin{vmatrix}a_2 & a_3\\ a_1 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & a_2\\ a_0 & a_1\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\\ 0 & a_0 & a_1\end{vmatrix}}$$

$$-\frac{a_0^4\begin{vmatrix}a_2 & a_3 & a_4\\ a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\\ 0 & a_0 & a_1\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3 & a_4\\ a_0 & a_1 & a_2 & a_3\\ 0 & a_0 & a_1 & a_2\\ 0 & 0 & a_0 & a_1\end{vmatrix}}$$

$$-\quad\dots\qquad(1)$$

This series converges rapidly when the ratio of the smallest root to every one of the other roots is small.

In the following note I propose to show that the method given by Whittaker can be used to determine the absolute values of all the real roots of an algebraic or transcendental equation. Supposing x_0 to be the smallest root and x_1 to be the next higher root we can write $x_1=x_0+h$, where h has to be determined. Substituting x_0+h for x in the above equation and writing $f(x)$ for the expression

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$$

and then expanding by Taylor's theorem, we obtain

$$f'(x_0)+\frac{1}{\underline{|2}}f''(x_0)h+\frac{1}{\underline{|3}}f'''(x_0)h^2+\frac{1}{\underline{|4}}f^{iv}(x_0)h^3+\frac{1}{\underline{|5}}f^{v}(x_0)h^4+\ldots=0$$

as an equation in h. By Whittaker's formula the value of h, which is the smallest root in absolute value, of this equation is given by the series

$$-\frac{f'(x_0)}{\frac{1}{\underline{|2}}f''(x_0)}-\frac{\left\{f'(x_0)\right\}^2\frac{1}{\underline{|3}}f'''(x_0)}{\frac{1}{\underline{|2}}f''(x_0)\begin{vmatrix}\frac{1}{\underline{|2}}f''(x_0) & \frac{1}{\underline{|3}}f'''(x_0)\\ f'(x_0) & \frac{1}{\underline{|2}}f''(x_0)\end{vmatrix}}$$

$$-\frac{\left\{f'(x_0)\right\}^3\begin{vmatrix}\frac{1}{\underline{|3}}f'''(x_0) & \frac{1}{\underline{|4}}f^{iv}(x_0)\\ \frac{1}{\underline{|2}}f''(x_0) & \frac{1}{\underline{|3}}f'''(x_0)\end{vmatrix}}{\begin{vmatrix}\frac{1}{\underline{|2}}f''(x_0) & \frac{1}{\underline{|3}}f'''(x_0)\\ f'(x_0) & \frac{1}{\underline{|2}}f''(x_0)\end{vmatrix}\begin{vmatrix}\frac{1}{\underline{|2}}f''(x_0) & \frac{1}{\underline{|3}}f'''(x_3) & \frac{1}{\underline{|4}}f^{iv}(x_0)\\ f'(x_0) & \frac{1}{\underline{|2}}f''(x_0) & \frac{1}{\underline{|3}}f'''(x_0)\\ 0 & f'(x_0) & \frac{1}{\underline{|2}}f''(x_0)\end{vmatrix}}$$

$$-\ \ldots \qquad (2)$$

where $f'(x_0), f''(x_0)\ldots$, mean $\dfrac{df(x_0)}{dx_0}, \dfrac{d^2f(x_0)}{dx_0^2}, \ldots$

Therefore the second real root of the original equation is given by

$$x_1 = \text{series (1)} + \text{series (2)}.$$

This process can obviously be extended to obtain all the higher roots one by one.

Since the value of the smallest root has to be obtained from an infinite series the determination of this root will always involve a certain amount of error. This error will be carried on in the determination of the next higher root and so on. So that after a certain stage this method will fail to give the value of the higher roots to any desired degree of approximation. This defect in the method can be partially avoided by replacing the equation in its original stage as well as in the different transformed stages for the determination of the higher roots by one which has undergone two or three Lobatchevsky-Graeffe operations, each of which replaces the equation operated on by an equation whose roots are the squares of its roots, so as to make the series for the determination of the roots more and more convergent and thereby minimise the error as far as possible in each step of the process.

As an example of this method, I have worked out the first three roots of the well known equation in Mathematical Physics :

$$\theta = \tan \theta,$$

or

$$\frac{2\theta^3}{\underline{|3}} - \frac{4\theta^5}{\underline{|5}} + \frac{6\theta^7}{\underline{|7}} - \frac{8\theta^9}{\underline{|9}} + \frac{10\theta^{11}}{\underline{|11}} - \ldots = 0,$$

or

$$1 - \cdot 1\theta^2 + \cdot 003571\theta^4 - \cdot 0000661 3\theta^6 + \cdot 00000075\theta^8 - \ldots = 0.$$

Considering this as an equation in θ^2, and using Whittaker's formula we get the smallest root of the equation in the form

$$\theta_0^2 = 10 + 5\cdot 5623 + 2\cdot 7727 + 1\cdot 1428 + \cdot 4834 + \ldots .$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get

$$\theta_0^2 = 20\cdot 20.$$

Therefore,

$$\theta_0 = \pm 4\cdot 49.$$

Then we seek the next higher root by the present method. The equation in h is

$$-\cdot 0174+\cdot 0009h-\cdot 00002h^2+\cdot 000004h^3-\ldots=0.$$

Using Whittaker's formula we get the lowest root of this equation in the form

$$h_0=19\cdot 3423+11\cdot 6242+4\cdot 4123+2\cdot 7213+1\cdot 0215+\ldots.$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get approximately

$$h_0=39\cdot 51.$$

Therefore,

$$\theta_1{}^2=59\cdot 71$$

or

$$\theta_1=\pm 7\cdot 72.$$

Again to find out the next higher root, we get the following equation in h

$$\cdot 0032-\cdot 00012h+\cdot 000016h^2-\cdot 000003h^3+\ldots=0.$$

By the help of Whittaker's formula, we obtain the lowest root of this equation in the form

$$h'_0=26\cdot 6732+17\cdot 3213+8\cdot 4221+4\cdot 1245+1\cdot 9234+\ldots.$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get approximately

$$h'_0=59\cdot 28.$$

Therefore,

$$\theta_2{}^2=118\cdot 99,$$

or

$$\theta_2=\pm 10\cdot 90.$$

Before I conclude I wish to thank Prof. S. K. Banerjee for having drawn my attention to the problem discussed in this paper and for the help which I have received from him.

Review[1]

CULLIS'S MATRICES AND DETERMINOIDS

University of Calcutta—Readership Lectures: Matrices and Determinoids, by C. E. Cullis, M.A., Ph.D., Vol. I, i–xii, pages 1–430 (1913); Price 21 Shillings; Vol. II, i–xxiv, pages 1–555 (1918); Price 42 Shillings, Cambridge; at the University Press.

Fuchs: To study *Modern Algebra* I'm most persuaded.
Meph: It was not my wish to lead thee astray.
But as concerns this Science, truly
'Tis difficult to avoid the empty form,
And should'st thou lack clear comprehension,
Scarcely the indices thou'll know apart.
'Tis safest far to trust but *one*
And built upon your master's formulas
On the whole—cling closely to your *symbols.*
Then, for the weal of research you may gain
An entrance to the formula's sure domain.
Fuchs: The symbol, it must lead to some result?
Meph: Granted. But never worry about results,
For, mind you, just when the results are wanting
A symbol at the nick of time appears.
To symbolic treatment all things yield,
Provided we stay in the general field.
Should a solution prove elusive,
Write the equation in determinant form.
Write what you please, but *never* calculate.
Symbols are patient and long suffering,
A single stroke completes the whole affair.
Symbols for every purpose do suffice.

Lasswitz, Kurd.

The Tragedy of Faust, &c., &c.

As Prof. C. R. Lanman of Harvard, speaking of the labours of a great oriental scholar, said :—

"Scholarship moves like the tides of the sea. It is started by some great celestial attraction, some force moving in an ecliptic high above the level world of letters (and of Science) and with gathering it comes to its flood."

[1] It is regretted that space cannot be found for the whole of this excellent review in this number of the Bulletin. It is proposed to publish the same in two instalments in successive issues of the Bulletin.—S. K. B.

Thanks to the enlightened interest and leadership of the Hon'ble Sir Asutosh Mookerjee, Kt., M.A., D.L., the President of the Post-Graduate Councils of Teaching in Arts and Science and to the band of devoted workers his genius has collected, the Calcutta University has already moved to an assured place in the world of letters and Science.

Foremost among the mathematicians working in the field of Pure Mathematics is Professor Charles Edmund Cullis, M.A. (Cantab.), Ph.D. (Jena), formerly Smith's Prizeman of the Cambridge University and Fellow of Gonville and Caius College, Cambridge, now Hardinge Professor of Mathematics in the University of Calcutta. And foremost among the mathematical productions in that University are: (1) A. R. Forsyth's Lectures introductory to the Theory of Functions of two complex Variables and (2) the remarkable work of which a review is attempted here. That work is an amplification of a course of lectures given for the University of Calcutta in the winter session of 1909-10. The student of mathematics comes across, frequently, in modern analysis, the term "matrix" and the fundamental laws of a matrix theory, but no-where does he find, search however he may, a *complete* and *consistent* calculus of matrices. "To give a satisfactory answer then to the frequently propounded question 'what is a matrix?'; it seemed advisible to Dr. Cullis to commence with some account of the Theory"; the result was that a decent volume of 430 pages was published in 1913, giving the most fundamental portions of the theory and concluding with the solution of any system of linear algebraic equations which is treated as a special case of a matrix equation of the first degree. The contents of the first volume can be broadly divided into two parts, *viz.*, the Theory of Determinoids and the Theory of Matrices. The theory of determinants has hitherto held the sway and Sir Thomas Muir, the learned author of the two well-known volumes on the History of that Theory, has expressed a doubt (vide Mathematical Gazette for December, 1913) whether the "Theory of Determinoid" as developed by Dr. Cullis will contribute to the advancement of science to the same extent as its fellow-subject Matrices or its prototype Determinants. But the "determinoid" stands in the same relation to the *rectangular* matrix as the "determinant" stands to the *square* matrix and surely, it is incumbent upon a mathematician to give, a scientific treatment, of the *most general* case of the *rectangular* matrix and its *content*, the determinoid. The aim of the author has been to construct and

develop the theory with a view to its extensive *applications* in Algebra, Geometry, Applied Mathematics, Vector Analysis and the Theory of Invariants. But the theory alone has been so engrossing in its character that the *applications* have had to be put further back. The author need however make no apology for, nor the mathematical world should mind, the delay, We should recollect Dr. Hobson's correct summing-up of the position of a true mathematician and the appraisement of his work. "Any attempt," says he, "to discourage perfectly untrammelled research in those parts of the subject that are most remote from *practical* interests, or that show least promise of fruitful *application* in other branches of science, would not only be a *vital* blow to mathematics as an evergrowing science, but would ultimately impair its efficiency as an *instrument*. Mathematics can only flourish if it has full *autonomy*. The nature and direction of its future applications in pure and applied science can never be fully foreseen. It is, however, extremely probable that the services it will render in the future will cover an even greater range than in the past, provided it is allowed to fit itself for rendering such services by according it full opportunity to develop itself in accordance with its own nature."

In taking stock of a recent work (Dickson's History of the Theory of Numbers), Professor Lehmer very appropriately says, "In these days when 'pure' science is looked upon with impatience, or at best with good-humored indulgence the appearance of such a book will be greeted with joy by those of us who still believe in mathematics for mathematics' sake." The reviewer is in good company, when he expresses his firm faith that "no great headway will ever be made in any science, least of all in Mathematics, by those who are always looking for the penny." He is also firm in the faith that as the years go by, the Matrix Theory (like the Theory of Numbers) will become increasingly popular, and works like Dr. Cullis's will contribute in a large measure, to render that theory eminently attractive. The reviewer would proceed now to the contents of that work.

THE DETERMINOID.

The first volume consists of a preface, eleven chapters and a complete and useful index. Each chapter is proceded by a summary of its contents—a new feature. It contains "the *foundations* of a Calculus of Matrices in which the oparations are, addition, subtraction

or multiplication and the result of performing any number of these operations with any rectangular matrices whatever is always a completely determinate matrix. It also contains an account of the properties of the *determinoid* of a rectangular matrix which becomes the *determinant* of the matrix in the particular case when the matrix is square." The book is so remarkable, so elaborataly written, so logical in its methods and so much new ground has been broken that the reviewer does not feel that he needs offer an apology for entering into the *details* of each chapter and indicating some of the results arrived at.

Chapter I gives an introductory account of *rectangular* matrices, of determinoids and a description of various abbreviated notations, both for matrices and determinoids, used in the text. Dr. Cullis writes the rectangular matrix of m horizontal and n vertical, rows (notice that he drops the time-honoured term "column" and rightly, I think) as

$$A=[a]_m^n=\begin{bmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ a_{m1} & a_{m2} \quad a_{mn} \end{bmatrix}$$

and the determinoid of the matrix as

$$(a)_m^n=\begin{vmatrix} a_{11}\ldots a_{1n} \\ a_{12}\ldots a_{2n} \\ a_{m1}\ldots a_{mn} \end{vmatrix}=\det\ [a]_m^n.$$

Other matrices can obviously be derived from the parent matrix A by—

(*i*) A rearrangement, in *any manner*, of the vertical and horizontal *rows* among themselves without any row being entirely struck out. This is called a *derangement* of A.

(*ii*) Striking out some rows of one kind or of both kinds, leaving the others in the *same relative orders* as in the matrix A. The matrix thus left is called a *corranged minor* matrix of A. When such a matrix is derived by the striking out of only *one kind* of rows it is called a *simple* minor matrix.

And (*iii*) by striking out rows of one kind or of both kinds from a derangement of A. The residue is called a *deranged* minor matrix of A.

The *corranged* and *deranged* minor matrices and the derangements, of A constitute, in their totality, the *derived* matrices of A. A matrix is said to *contain* each of its minor matrices, deranged and corranged, thus giving significance to the "root" meaning of the word "matrix" and illustrating to some extent the remark of Sylvester (one of the great pioneers in the theory of matrices) that "a matrix is a rectangular array of terms, out of which different systems of determinants (determinoids) may be engendered as from the *womb* of *a common parent*; these cognate determinants (determinoids) being by no means isolated in their relations to one another but subject to certain simple laws of mutual dependence and simultaneous deperition." The object of the theory of matrices is to exhibit these laws and their consequences.

From the definition of derived matrices, the author passes on to the definition of *complete* and *incomlete* derived products of the *elements* of the parent matrix (Sub Art. 3) and to the definition of a *step* (backward or forward, horizontal or vertical) (Sub Art. 4), thus paving the way for the definition of a *determinoid* (Art. 3), *viz.*, that the determinoid of any matrix A, is the *algebraic* sum of *all* its *complete* derived products (a complete derived product being the product of the *largest possible* number of elements selected so that no two of them occur in the same vertical or horizontal row) when each product is subject to a *rule of sign*, the elements of the parent matrix being assumed, for present purposes, to be *scalar* numbers, and the *order* of the factors being immaterial. Dr. Cullis here introduces the term "affect" of a complete product which determines its sign. Thus, if α', β', γ', δ',... are the *sums* of the horizontal and vertical steps by which, counting from the leading element, we arrive *respectively* at each element α, β, γ, δ,... of a complete product $\alpha\beta\gamma\delta$...beginning with the original matrix and then passing on successively to the matrices obtained by striking out in turn the vertical and horizontal rows which cross at any particular element, the number $\omega=\alpha'+\beta'+\gamma'+\delta'+$...is called the *affect* of the product $\alpha\beta\gamma\delta$... and the sign to be attached to the latter is determined by $(-1)^{\omega}$. The question arises, is this rule of sign *self-consistent*? Dr. Cullis gives a complete proof, in small type, that the answer is in the affirmative. The determinoid and, it is proved afterwards, in Art. 13, page 48, the algebraic sum of all the affected derived products of order r when r is not greater than the efficiency of the matrix are

Affect of a product.

thus shown to be *definite* functions of the elements of the matrix. The ground is now prepared for giving some elementary properties of determinoids. They are—

Properties of a determinoid.

(*i*) The value of a determinoid is unaltered by interchanging its horizontal and vertical rows. Thus a determinoid can always be written with its long rows horizontal and two conjugate matrices (defined in Art. 5 but it appears to the reviewer that it might have been defined earlier, say, just after Art. 1 giving the definition of matrices of various orders) have their determinoids equal in value, although they, the matrices themselves, are generally different.

(*ii*) A determinoid is a homogeneous *linear* function of the elements of any *long* row.

(*iii*) If each element of a *long* row of a determinoid is split up into the sum or difference of two other elements, the determinoid is split up into the sum or difference of two other determinoids. This can be easily generalized.

(*iv*) Multiplying every element of any long row by a number κ is equivalent to multiplying the determinoid by κ.

(*v*) If any two *long* rows are interchanged, the determinoid is changed in sign but is unaltered in magnitude. Thus, when the elements of two long rows are identical, the determinoid vanishes and the value of a determinoid is unaltered by adding to the elements of any *long* row the corresponding elements of any other long row, each multiplied by the same number. (Dr. Cullis uses in Sub Art. 5 of Art. 5, page 20, the word "quantity" which should more appropriately I think be "number").

Chapter I is concluded with the pertinent remark that "as regards *long* rows, the properties of determinoids are closely analogous to those of determinants which are of course, a special class of determinoids."

Chapter II deals with the *affects* of the elements and derived products of a matrix or determinoid. At the outset it is discovered that

Affects of the elements of a matrix.

the *affects* of the *elements* of a matrix have not been formally defined. This ommission has been made good in small type in the preface to the first volume as Art. 5a. The *affect of the element* $a_{x,y}$ of the matrix $[a]^{n}_{m}$ is defined to be the number ω given by the equation $\omega=(x-1)+(y-1)$. $(x-1)$ is called the vertical affect and $(y-1)$, the horizontal

affect of a_{xy} the total affect or simply, the affect of a_{xy} being their sum. This being premised it is natural to define the affect, ω, of a derived product of order r, *viz.*, $P=z_1z_2\ldots z_k\ldots z_r$, to be $\omega_1+\omega_2+\ldots\omega_k+\ldots\omega_r$, where ω_k, is the affect of the element z_k in matrix A_{k-1} obtained from the original matrix A by the *successive* deletion of $k-1$ vertical and horizontal rows passing through the previous $k-1$ elements, $z_1, z_2,\ldots z_{k-1}$. This is, in substance, the rule of signs given in Art. 3 of Chapter I when applied to any derived product of a matrix or determinoid instead of only to the *complete* derived products of the latter. The splitting-up of ω_k into two elements ω_k' (the vertical affect) and ω_k'' (the horizontal affect) is obvious and we get $\omega=\omega'+\omega''$ where $\omega'=\Sigma\omega_k'$ and $\omega''=\Sigma\omega_k''$. Clearly, in practice, the determination of affects of derived products by the striking out of rows and the counting of steps is inconvenient and practical rules are required for simplifying the process. This desideratum is attained in Sub. Art. 3 of Art. 8 in which rules are given which can be worked by the mere inspection of the relative *positions* of the elements with respect to horizontal and vertical rows of the matrix and in the product itself. In Sub. Art. 4 of Art. 8, the important fact is deduced that the affect of a derived product when it is *extended* or *completed*, (extension and completion having been defined in Art. 6) is not changed; whence it follows that the affects of all derived products can be reduced to the determination of the affects of complete derived products. Sub. Art. 2 of Art. 8 gives simple rules for doing this. In reading this sub-article, however, it seems to the reviewer, that the expression "horizontal affects" in line 5 of the second paragraph should be "vertical affects" and that the expression "vertical affects" in line 6 should be "horizontal affects." Similar changes would also be necessary in the sentence of this paragraph which follows "vertical affects" in line 6. Further down also, at the bottom of the page, (Ex. iv) the expression "vertical suffixes" should be replaced by "horizontal suffixes."

Changes in the affect of a derived product caused by the interchange of two *consecutive* parallel rows and, more generally, of *any* two parallel rows of a matrix, are exhaustively dealt with in Arts. 9 and 11 of this Chapter and copiously illustrated.

Changes in the affect of a derived product by the interchange, in the *product*, first of *consecutive* suffixes of the same kind, and, secondly, of *any* two suffixes of the same kind, are similarly dealt with in Arts. 10 and 12. Art. 13 brings prominently to notice

(as already referred to above) the *invariance* of the *sign* of a derived product—complete or incomplete. We are now ready for an important operation in the theory, *viz.*, the reduction of any derived product of a matrix M to a *leading product* (the reviewer misses the definition of a "leading product") by forward moves, the word "move" with the qualifications "vertical and horizontal," "forward" and "backward" being defined (Art. 14). In the illustrative example (*iii*) below the Article, it is logically deduced that "the *determinoid* of a square matrix," as developed in Dr. Cullis's Calculus, is identical with the *determinant* of that matrix, as ordinarily defined.

We have hitherto been concerned with changes, so to speak, due to the motion of *translation* of entire rows. Art. 15 introduces changes due to the motion of *rotation* through two right angles in the rectangular array called the matrix, round three mutually perpendicular axes, two lying in the plane of the rectangle bisecting the opposite sides and the third perpendicular to that plane, through its centre. The results obtained are important for future use and, for immediate use in Art. 16, which gives the results of the three kinds of *inversion* of the orders of arrangement of the rows of a determinoid, $(a)^n_m = \Delta$ They are as follows :—

I. When the *long* rows are inverted :—

$$\Delta' = (-1)^{\frac{m(m-1)}{2}} \times \Delta.$$

II. When the *short* rows are inverted :—

$$\Delta' = (-1)^{m(n-m) + m\frac{(m-1)}{2}} \times \Delta.$$

III. When *both* sets of rows are inverted :—

$$\Delta' = (-1)^{m(n-m)} \times \Delta.$$

Sequences and their affects

Chapter III dealing with "*sequences* and the affects of derived *sequences*" may well be characterized as a 'simplifying' chapter and opinion may be divided as to whether it should not have found an earlier place in the work. A *sequence* is any *linear* arrangement of *elements* which are letters or numbers, such as :—

$$A = [a_1 a_2 \ldots a_n].$$

It can, of course, be regarded as a matrix with only one long row and it is possible to adapt the definitions and some of the deductions in regard to the matrix $[a]_m^n$ in the previous chapters, *mutatis mutandis* to *sequences.* But Dr. Cullis with his logical acumen, works out a new chapter, an interesting one in Combinatory Analysis. Specially valuable are the results obtained, in Art. 19, concerning the affects of derived sequences which are of use in the following chapters, in view of the remarkable result obtained at the end of the chapter, in Art. 22, *viz.*, the reduction of the affects of *derived products* (of a matrix) to the affects of *sequences.* Thus, "all properties of the *affects* of derived products of a matrix or determinoid, can be deduced from the properties of affects of *sequences.*" This is, indeed, a great simplification.

Having dealt successively with the affects of *derived* products of a fundamental matrix or determinoid and the affects of *derived* sequences of a fundmental sequence, the author naturally passes on, in Chapter IV, to affects of derived matrices and derived determinoids. Obviously a definition of the *affect* of a derived matrix (which would be applicable to a derived determinoid also), is needed and this is given in Art. 23 in the most natural manner. If $A=[a]_m^n$ be the fundamental matrix and $B=[a_{xy}]_u^v$ be any *derived* matrix (page 3) of A, then, writing B in the expanded from in double-suffix notation as

$$\begin{bmatrix} a_{x_1 y_1} & a_{x_1 y_2} \cdots & a_{x_1 y_r} \\ a_{x_2 y_1} & a_{x_2 y_2} & a_{x_2 y_r} \\ \cdots & \cdots & \cdots \\ a_{x_u y_1} \cdots & \cdots & a_{x_u y_r} \end{bmatrix}$$

the affect of the derived matrix B in the fundamental matrix A is *defined* to be the quantity ω given by the equations :—

ω'=the affect of the *sequence* $[x_1 x_2 \ldots x_u]$ in the *sequence* $[1\ 2 \ldots m]$ and is called the vertical affect of B in A; ω'=affect of the *sequence* $[y_1 y_2 \ldots y_r]$ in the *sequence* $[1\ 2 \ldots n]$ and is called the horizontal affect of B in A.

$$\omega=\omega'+\omega''.$$

But, if the definition is to be useful, it must make us independent of any *particular* notation used in writing the derived matrix.

Dr. Cullis therefore gives us two more differently worded definitions on page 88.

As in the ca e of derived products and derived sequences, so also in the case of derived matrices or determinoids, rules are given for the *extension* and *completion* of matrices derived from a fundamental matrix and it is shown that the *affect* of a derived matrix is not altered by *extension* or *completion* (Art. 24). Art. 25 is concerned with several important theorems relating to the *affects* of derived matrices which are, naturally, analogous to the theorems in Art. 19 relating to the affects of derived sequences. In Art. 26 *complementary* derived matrices are defined and the two theorems VIIIa and VIIIb of Art. 19 regarding complementary *sequences* are extended to complementary *matrices* or *determinoids*.

Epaxnsion of a determinoid.

The reader is now fully equipped for dealing with the *expansion of a determinoid* which is dealt with in Chapter V. Three methods are first given of such expansion, *viz.*—

(1) Expansion in terms of the elements of any *given* long row (Art. 27).

(2) Expansion in terms of the simple minor determinants (Art. 30).

And (3) expansion in terms of the simple minor *determinants* of a given *long cut* minor as defined in Art. 31 (Art. 32).

Art. 28 gives only a glimpse of reciprocal matrices and determinoids and the three examples appended to the article in small type give some relations which are obvious consequences of the definitions and the first method of expansion.

Properties of short rows.

Art. 29 deals with the properties of the *short rows* of a determinoid. It is shown that the determinoid is in general, a *non-homogeneous* linear function of the elements of any selected *short* row.

It is also shown that in the case of the determinoid $(a)^n_m$ where n is $> m$:

(i) we can expand the determinoid in terms of the elements of any given $n-m+1$ short rows,

(ii) the value of a determinoid is not altered by *post-fixing* an *additional* short row of zeros *after* the existing short rows,

and (iii) the value is multiplied by $(-1)^m$ by *prefixing* such an additional short row.

All these results are consequences of the general theory of Art. 30 in which the expansion of a determinoid in terms of its *simple minor determinants* is effected. This is noticed below :—

Let A be any determinoid with m long rows and n short rows. Let D_1 D_2. . D_t. . . D_r be the *determinoids* which can be formed from A by the omissions of short rows. These are the *corranged* simple minors of A which are determinants. Their number is $u=\binom{n}{m}$. Then, it is easily established,

$$A=(-1)^{\omega_1}D_1+(-1)^{\omega_2}D_2+\ldots(-1)^{\omega_u}D_u.$$

where ω_t is the affect of D_t in A.

This expansion, it is then proved, remains true when the simple minor determinants are *deranged* any way. So that "A determinoid is the algebraical sum of all *its* distinct simple minor determinants, *corranged* or *deranged,* when each of these determinants has the sign determined by its affects in the original determinoid." This is an important result.

Art. 31 gives, (*i*) the classification of simple minor *determinoids* into long-cut and short-cut minors (*ii*) the definition of the *reduced order* of the simple minor determinoid and (*iii*) the definition of of *superior* and *inferior* minors.

These definitions might, as they cover matrices also, have come earlier say, in Chapter I but Dr. Cullis has obviously a plan of his own according to which he succeds in making each chapter self-contained.

Generalization of Laplace's development of a determinant.

Art. 32 gives the expansion of a determinoid in terms of the simple minor determinants of a given *long-cut* minor matrix of the matrix of the derterminoid Setting

$$\Delta=[a]_m^n=\det[a]_m^n.$$

$$U=[a_{\kappa 1}]_u^n$$

$B=(b)_n^u$ =a corranged determinant of order u formed from u by striking out $n-u$ *short* rows.

$C = (c)^{\sigma}_{\rho}$ = a corranged determinoid complementary to B in Δ.

ω = affect of B in Δ.

We obtain

$$\rho = m - u \quad \sigma = n - u$$

$$\Delta = \Sigma(-1)^{\omega} BC.$$

The number of terms in the sum is obviously $\binom{n}{u}$.

The proof given is lucid and logical and defining the cofactor of any minor determinant B (corranged or deranged) of a determinoid (or matrix) Δ, to be the *corranged* minor determinoid *complementary* to B in Δ and affected with the sign determined by the affect of B in Δ, Dr. Cullis arrives at what may be regarded as a *generalization* of Laplace's deveolopment of a determinant, *viz.*, "A determinoid is the algebraical sum of all the products which are obtained when every distinct simple minor determinant of a given *long-cut* minor matrix is multiplied by its cofactor." Illustrative examples follow and the student of the calculus of determinoids will do well to remember the interesting result obtained in exapmple V (page 120), *viz.*, if Δ is the determinoid

$$\begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \\ 1 & 1 \ldots\ldots 1 \end{vmatrix}$$

then $\Delta = 0$ or $(a)^{n}_{m}$ *i.e.*, the determinoid of $[a]^{n}_{n}$, according as $n - m$ is even or odd.

Below the examples on page 120 is given an important *theorem*, which is not given in bold black type as other theorems are. Moreover, the theorem concerns *similar* matrices of which no mention is made in previous pages. It is not till we come to Art. 39, page 153 that we find *similar* matrices defined.

Art. 33 gives a striking theorem involving an important function. The theorem is $S_n = \Delta \times Q^{n-m}_{n-m}$, where S_n is the algebraical sum of all the corranged superior simple minor determinoids of reduced order n derived from a fundamental determinoid Δ with m long rows and n short rows, when each minor has its sign determined by its affect in Δ and $Q^{n-m}_{n-m} = \Sigma(-1)_k$, the values of k being the *affects* of all possible corranged minor sequences of $n-m$ elements derived from a fundamental sequence of $n-m$ elements. The function, then, of importance is Q^{n}_{m} and various properties of this are discussed in Art. 34. This is printed in small type, probably, because it is somewhat of a digression from the main topic under discussion. This function had, it may be noted, been previously discussed by Metzler in Vol. XXII of the American Journal of Mathematics as $\phi(n,m)$ and it is so interesting that we may notice some of the properties worked out by Dr. Cullis :—

In sub-article 2 (*ii*) of Art. 34 it is proved that

$$Q^{n}_{m} = Q^{n}_{n-m} \quad ; \; m \not> n.$$

Sub-article (3) gives formulae of *reduction* for Q^{n}_{m}. They are

$$Q^{m+r}_{m} = Q^{m+r-1}_{m-1} \quad + (-1)^{m} \; Q^{m+r-2}_{m-1} \quad + (-1)^{2m} Q^{m+r-3}_{m-1}$$

$$\ldots + (-1)^{rm} \; Q^{m-1}_{m-1}$$

$$= Q^{r-1}_{0} \quad + (-1)^{1} Q_{1} + \ldots (-1)^{m} \; Q^{m+r-1}_{m}.$$

With the help of these formulae of reduction a tabular representation of the values of the function for different values of m and n is given in sub-article 4. And in sub-article (5) it is shown that

$$Q^{2n}_{2m} = Q^{2n+1}_{2m+1} = Q^{2n+1}_{2m} = \binom{n}{m}$$

and that

$$Q^{2n}_{2m+1} = 0.$$

After discussing in Art. 35 and 36 algebraic sums of (inferior) short-cut and (inferior) long-cut simple minor determinoids of given reduced order, obtained from the fundamental determinoid, the author gives in Arts. 37 and 38 theorems which *generalize* the preceding results. These two articles contain an investigation of the algebraical sum of the products obtained by multiplying each affected simple minor determinoid of given reduced order, of a given simple minor matrix of the fundamental determinoid, by its *c rranged* complement or cofactor. In case I of Art. 38, *i.e.*, when both the minor determinoid of given reduced order, A_{v} and its cofactor, A_{m-v}^{n-v} are *superior* simple minors of the respective matrices, the sum is a certain *numerical multiple, viz.*, $\pm\, Q_{v-v}^{m-n}$ of the fundamental determinoid. This clearly includes the result of Art. 32 and Dr. Cullis is justified in calling the result *"a still further generalization of Laplace's development of a determinant."* Chapter V concludes with seven pages of closely written small type giving the formal proofs of the three cases of Art. 38, the concerete examples, illustrating these three cases and just preceding the proofs, being specially valuable, as leading to a right understanding of the proofs which are not easy to follow.

(*To be continued.*)

A. C. BOSE.

CALCUTTA MATHEMATICAL SOCIETY

REPORT FOR THE YEAR 1919.

1. The gentlemen named below were elected officers and other members of the Council for the year 1919 :—

President :

The Hon'ble Justice Sir Asutosh Mookerjee, Swaraswati, Kt., C.S.I., M.A., D.L., D.Sc., Ph.D., F.R.S.E.

Vice-Presidents :

The Hon'ble Mr. Mahendranath Ray, C.I.E., M.A., B.L.
Dr. C. E. Cullis, M.A., Ph.D.
Dr. Syamadas Mookerjee, M.A., Ph.D.

Treasurer :

Rai Bahadur Abinaschandra Bose, M.A.

Secretary :

Dr. Sudhansukumar Banerjee, D.Sc.

Council :

Dr. D. N. Mallik, D.Sc., F.R.S.E.
Dr. S. C. Bagchi, LL.D., Bar-at-Law.
C. V. Raman, Esq., M.A.
Satishchandra Bose, Esq., M.A.
Manmathanath Ray, Esq., M.A., B.L.
Narendrakumar Majumdar, Esq., M.A.
Phanindralal Gangooly, Esq., M.A., B.L.
Surendramohan Gangooly, Esq., M.Sc.
Bibhutibhusan Datta, Esq., M.Sc.

2. During the year 1919 eight meetings were held against five in 1918 and six in 1917 and 35 papers were communicated against 17 in 1918 and 24 in 1917.

3. Four issues of the Bulletin (*viz.*, Vol. IX, No. 2; Vol. X, Nos. 1, 2 and 3) containing 22 original papers, 2 reviews, 4 obituary notices and miscellaneous notes have been published in 1919 against two issues of the Bulletin containing 11 original papers and one review in 1918 and two issues of the Bulletin containing 10 original papers and two reviews in 1917. The original idea of the Society of publishing four numbers of the Bulletin in one year has been for the first time realised this year. The four issues of the Bulletin have been published quarterly in March, June, September and December. Each number of the Bulletin consists of nearly eight forms, that is to say, 64 pages and the four together forms a decent volume exceeding 250 pages.

4. The year opened with the happy termination of the terrible war and an exchange of the warmest felicitations with the Societé Mathematique de France and other learned societies of Europe. It is to be hoped that the termination of the war and the inauguration of the reforms will remove at least partially some of the serious difficulties that stand in the way of scientific research in this country.

5. During the year the world has lost a number of distinguished mathematicians. Professor Ulisse Dini of the University of Pisa, Professor G. Cantor of the University of Halle, Professor Maxime Bocher of the University of Harvard, Professor O. Henrici of the City and Guilds Technical College and Professor Liapounoff of the University of Petrograd have left their marks of impression as some of the greatest thinkers of the world. By the death of the Rt. Hon. Lord Rayleigh, the world has lost a great scientist and the Society a sincere well-wisher. Lord Rayleigh was an Honorary Member of the Society since its foundation and visited India some years ago. His death will be considered as a personal loss by several Indian workers for not a few of them have received personal encouragement from him. By the death of Mr. Chandra Shekhar Sircar, the Society has lost a great benefactor and a life member. The Society has also to record the loss of Principal Ramendrasundar Trivedi who was for many years a member of the Society and one of the greatest educationists of Bengal.

6. During the year the Society has received letters from several distinguished savants of Europe and America expressing their appreciation of the work done by the Society and extending their

co-operation and fellowship for which the Society offers its best thanks. The Society has entered into new exchange relations with several learned Societies of Europe, America, Japan and Australia. The present exchange list of the Society contains more than fifty different names. The Bulletin of the Society is now being subscribed by some of the Universities of Europe and America.

7. Arrangements have been made for the regular review of all the papers published in the Bulletin in the Revue Semestrielle des Publications Mathématiques. The Society is also in correspondence with the Editors of Fortschritte der Mathematik for review in that journal of all the papers published in the Bulletin.

8. The Calcutta Mathematical Society as it now stands after ten years of active and continuous existence may be said to have been firmly established. A bright future of the Society is assured under the benevolent care of its distinguished first President who has always given to the Society ungrudgingly and cheerfully a portion of his most precious time to the great encouragement of the members of the Society. There has seldom been a meeting of the Society in which his inspiring presence has been missed. Under his fostering care a bright progeny of young mathematicians are fast springing up in Bengal. The output of mathematical research in Bengal is yearly increasing in quality and quantity, so much so, that the Society will have ere long to seriously consider the issue of its Bulletin once in two months, if not oftener.

9. The best thanks of the Society are due to the authorities of the Calcutta University Press for publishing the Bulletin on behalf of the Society almost free of charge and to the Superintendent of the Press for expediting the printing of the Bulletin which has enabled the Society to issue four numbers in the year under review.

10. Twenty-three ordinary members and five honorary members were elected in 1919 against 17 ordinary members in 1918 and 7 in 1917. There are altogether about 160 ordinary members and 25 honorary members.

APPENDIX A

The following papers were read before the Calcutta Mathematical Society during the year 1919 :—

1. Dr. Syamadas Mookerjee :—"Normals and Cyclic Points." (Vol. X, No. 2).

2. Prof. Sudhansu Kumar Banerji :—"On Surface Waves and Tidal Waves near a Promontory." (Vol. X, No. 1).

3. Mr. Mohitmohan Ghosh :—"The Mathematical Theory of Diffraction by an Elliptic Aperture."

4. Mr. Bhupendra Chandra Das :—"On the formation of optical images by a diffracting boundary." (Vol. X, No. 3).

5. Dr. Syamadas Mookerjee :—"On the generalization and correlation of certain theorems in hyperbolic geometry."

6. Mr. Sasindra Chandra Dhar :—"On a certain integral equation and on Joachimsthal's attraction problem." (Vol. X, No. 3).

7. Mr. Bijanchandra Dutt :—"On the steady motion of a viscous fluid due to the rotation of two rigid bodies about arbitrary axes." (Vol. X, No. 1).

8. Mr. Sasadhar Das Gupta :—"On some cases of tidal oscillations in canals of variable section." (Vol. X, No. 2).

9. Mr. P. N. Dutt :—"Mathematics in every-day life."

10. Dr. Syamadas Mookerjee :—"The Rectangular Pentagon in Hyperbolic and Spherical Trigonometry."

11. Prof. C. V. Raman :—"A note on the Doppler effect in the molecular scattering of radiation."

12. Prof. C. V. Raman :—"On the diffraction of light within the human eye."

13. Prof. Sudhansu Kumar Banerji :—"On the diffraction of light by a dielectric wedge." (Vol. X, No. 4).

14. Mr. Nikhilranjan Sen :—"On the application of discontinuous integrals to the determination of the potentials of heterogeneous incomplete ellipsoids." (Vol. X, No. 3).

15. Mr. Satyendranath Basu :—"On the integration of the stress equations in elasticity." (Vol, X, No. 2).

16. Mr. Haripada Datta :—"On some properties of natural numbers." (Vol. X, No. 4).

17. Prof. C. E. Cullis :—"Evaluation of the product matrix in any commutantal product of simple hemipteric matrices."

18. Prof. C. V. Raman :—"On the form of the internal fractures produced by impact."

19. Prof. Sudhansu Kumar Banerji :—"On a class of ellipsoidal harmonics." (Vol. X, No. 2).

20. Mr. Nripendranath Sen :—"Liquid motion inside a rotating elliptic quadrant."

21. Mr. Sisirkumar Mitra :—"On the diffraction figures due to a heliometer."

22. Mr. Satyendranath Basu :—"On the herpolhode."

23. Mr. Bholanath Pal :—"On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and $\frac{d}{d\mu} P_n^m(\mu)=0$ regarded as equations in n, Part II." (Vol. X, No. 3).

24. Mr. Abanibhusan Datta :—"On a geometrical treatment of the problem diffraction by a Cone." (Vol. X, No. 4).

25. Mr. Bhupatimohan Sen :—"A note on the deformation of surfaces." (Vol. X, No. 4).

26. Prof. Sudhansukumar Banerji :—"On Spherical Waves of Finite Amplitude."

27. Mr. Nikhilranjan Sen :—On the vibration of an elastic sperical shell in agaseous medium."

28. Mr. Bholanath Pal :—"On the motion of an elongated spheroid in a viscous fluid."

29. Mr. Nripendranath Chatterji :—"On some defects in the existing methods of solving algebraic equations by radicals, Part I."

30. Mr. Bibhutibhusan Datta :—"On the stability of a hollow vortex in a compressible fluid."

31. Prof. C. E. Cullis :—"The general commutant of any two simple square antecanonicants."

32. Prof. Sudhansukumar Banerji :—"One some peculiarities of the free oscillations of a gas within a rigid ellipsoidal envelope." (Vol. X, No. 3).

33. Mr. Surendranath Sen :—"The double-origin angular co-ordinates."

34. Mr. Nripendranath Chatterji :—"On some defects in the existing methods of solving algebraic equations by radicals, Part II."

35. Mr. S. M. Kamalkar :—"Note on a process in factorisation."

APPENDIX B

Treasurer's Statement for the year 1919.

[Audited.]

RECEIPTS.	Rs.	A.	P.
Opening Balance	322	13	0
Receipts from annual contributions, admission fees and sale of publications	1,067	0	0
TOTAL ..	1,389	13	0

DISBURSEMENTS.	Rs.	A.	P.
Purchase of books and periodicals	70	12	0
Book binding	119	8	0
Establishment	428	0	0
Payment of outstanding bills of the previous year	5	8	0
Miscellaneous, including charges for printing of notices, preparation of blocks, expenses of meetings, postage and stamps and other contingencies	289	13	6
Closing Balance	476	3	6
TOTAL ...	1,389	13	0